LECTURE :3 Forced Vibration

Let a particle of mass m executing damped SHM be subjected to an external simple harmonic force of constant amplitude and constant frequency, Fcoswt.

Forces acting on the particle:

- Force of restitution/ restoring force, -sx,
- Resisting force, $-k\frac{dx}{dt}$
- External force, $Fcoswt$.

Therefore, the equation of motion,

$$m\frac{d^2x}{dt^2} = Fcoswt - k\frac{dx}{dt} - sx$$

$or, \frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + w_0^2x = \frac{F}{m}coswt$---(1)

Where, $2b = \frac{k}{m}, w_0^2 = \frac{s}{m}, \ w_0 = natural\ frequency, b = damping\ constant$

General solution of (1) =complementary function(CF) + Particular integral(PI)

To get CF, put RHS=0

i.e $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + w_0^2x = 0$--(2)

Therefore, CF = $Cc^{-bt}cos\left(\sqrt{w_0{}^2 - b^2}\, t - \theta\right)$, C , θ are constants.

To get PI, put RHS $=\frac{F}{m}e^{jwt}$

i.e $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + w_0^2x = \frac{F}{m}e^{jwt}$--------------------------------------(3)

Note: If ext force is $Fcoswt$, then PI= real part of soln., x_r

If ext force is $Fsinwt$, then PI= imaginary part of soln., x_i

Take $x = s_0e^{j\omega t}$

Substituting in (3) we get,

$$s_0 = \frac{F/m}{w_0^2 - w^2 + 2bjw}$$

Take, $w_0^2 - w^2 = R\cos\emptyset$, $2bw = R\sin\emptyset$

$$s_0 = \frac{F/m\, e^{-j\emptyset}}{\left[\sqrt{\left(w_0^2 - w^2\right)^2 + 4b^2w^2}\right]}$$

$$\text{PI} = s = s_0 e^{j\emptyset} = \frac{F/m\, e^{-j\emptyset}}{\left[\sqrt{\left(w_0^2 - w^2\right)^2 + 4b^2w^2}\right]}$$

Complete solution:

$$x = Cc^{-bt}\cos\left(\sqrt{w_0{}^2 - b^2}\, t - \theta\right) + \frac{F/m\, e^{-j\emptyset}}{\left[\sqrt{(w_0^2 - w^2)^2 + 4b^2w^2}\right]}$$

For driving force Fcoswt, PI = $\frac{F\,coswt}{w|z_m|}$

where, $|z_m| = \left[\left(\frac{mw_0^2}{w} - mw\right)^2 + 4b^2m^2\right]^{1/2}$

Complete solution:

$$x = Cc^{-bt}\cos\left(\sqrt{w_0{}^2 - b^2}\, t - \theta\right) + \frac{F\cos(wt - \emptyset)}{[w|z_m|]}$$

C and $\emptyset$ to be determined from initial condition of x and $\frac{dx}{dt}$ at t = 0.

Initially , both CF and PI are operative, their resultant contribution give irregular motion. After sometime CF dies off with decay const b. Finally only PI exits.

PI : Steady state solution

CF : transient solution

In steady state particle oscillates with const amplitude $\frac{F}{w|z_m|}$, with period of impressed force, but lags the impressed force by angle $\emptyset$.