How do we extract information from the wave function? How do we predict the outcome of a measurement?

If we know ψ(x,t) we can make predictions about position measurements.

But what if we are not interested in the position, but in some other measurable quantity, such as the momentum or the energy? How does the wave function let us make predictions about those measurement?

We have to "operate" on the wave function. Every measurable quantity or observable is associated with an operator, something we have to do to the wave function with that operator.

- The operator for the momentum p_x is $(\hbar/i)\partial/\partial x$. It is a differential operator. We have to take the partial derivative of the wave function with respect to x and then multiply by ħ/I to know the momentum.

- The operator for the energy E is $i\hbar\partial/\partial t$. It also is a differential operator. We have to take the partial derivative of the wave function with respect to t and then multiply by iħ to know the energy.

- The operator for the position x is x. We have to multiply the wave function by x to know the position.

If the operator for a particular observable O operates on a wave function ψ(x,t), and the result of this operation is the wave function ψ(x,t) multiplied by a constant, then the wave function is said to be an eigenfunction of the operator and the constant one of its eigenvalues. A measurement of the observable at time t will for certain yield the eigenvalue. There will be no uncertainty about the outcome of the measurement.

Example:

Assume a free particle has the wave function $\psi(x,t) = A\exp(i(k_1x - \omega_1t))$.

Q. Is ψ(x,t) an eigenfunction of the momentum operator p_x?

The operator for p_x is $(\hbar/i)\partial/\partial x$.

$\partial\exp(i(k_1x - \omega_1t))/\partial x = ik_1 \exp(i(k_1x - \omega_1t))$

$(\hbar/i)\partial A\exp(i(k_1x - \omega_1t))/\partial x = \hbar k_1 A\exp(i(k_1x - \omega_1t))$

$(\hbar/i)\partial\psi(x,t)/\partial x = \hbar k_1 \psi(x,t)$

ψ(x,t) is an eigenfunction of the momentum operator. Its eigenvalue is $\hbar k_1$.
If we make a measurement, we will for sure measure the momentum to be $p_1 = \hbar k_1$.

Q. Is ψ(x,t) an eigenfunction of the energy operator E?

The operator for E is $i\hbar\partial/\partial t$.
$\partial \exp(i(k_1x - \omega_1t)))/\partial t = -i\omega_1 \exp(i(k_1x - \omega_1t))$
$i\hbar\partial A\exp(i(k_1x - \omega_1t))/\partial t = \hbar\omega_1 A\exp(i(k_1x - \omega_1t))$
$i\hbar\partial\psi(x,t)/\partial t = \hbar\omega_1 \psi(x,t)$
ψ(x,t) is an eigenfunction of the energy operator. Its eigenvalue is $\hbar\omega_1$.
If we make a measurement, we will for sure measure the energy to be $E_1 = \hbar\omega_1$.

Q. Is Is ψ(x,t) an eigenfunction of the position operator?

The position operator is x.
$xA\exp(i(k_1x - \omega_1t)$ is not equal to a constant times $A\exp(i(k_1x - \omega_1t)$.
ψ(x,t) is NOT an eigenfunction of the position operator.

If the operator for a particular observable O operates on a wave function ψ(x,t), and the result of this operation is NOT the wave function ψ(x,t) multiplied by a constant, then the wave function is NOT an eigenfunction of the operator and there is **uncertainty** about the outcome of a measurement. **The result of every measurement of an observable always will be one of its eigenvalues.** But if the wave function ψ(x,t) is NOT an eigenfunction of the operator then **all we can predict is the probability of measuring any of the possible eigenvalues**. We then can predict the **average value** of repeated measurements on identically prepared systems, but we cannot predict the outcome of an individual measurement.

Example:

- Find the average position <x> of a particle at t = 0 given the normalized wavefunction ψ(x,0).

  Note: The probability of finding the particle at time t in an interval Δx about the position x is equal to $|\psi(x,t)|^2\Delta x$.

  The average position <x> is the sum over all possible positions of the product x times $|\psi(x,t)|^2\Delta x$, i.e. the position times the probability of finding the particle at that position. $\langle x\rangle = \sum x_i |\psi(x_i,t)|^2\Delta x_i$
  [We really must evaluate that sum in the limit that Δx_i goes to zero.

We write $<x> = \sum_{\Delta x \to 0} x_i |\psi(x_i,t)|^2 \Delta x_i = \int_{-\infty}^{+\infty} x|\psi(x,t)|^2 dx$. The sum becomes an integral.]

Operations can change the information that you have about the particle and therefore change the wave function, or they can preserve it.

(When you do something to a wave function, you may change it in the process.)

- If the initial information about a particle came from a particular measurement, then the resulting wave function is said to be an eigenfunction of the operator associated with this measurement. If the operator associated with a different observable does not change this eigenfunction, then the two measurements are said to be **compatible**. We can know the value of both observable simultaneously. But if the operator associated with a different observable changes the eigenfunction of the first observable, then the two observables are **incompatible**. We cannot know the value of both observable simultaneously with arbitrary precision. The uncertainties in the values of both observables will be related by a generalized uncertainty principle.

- In quantum mechanics, a measurement of an observable yields a value, called an eigenvalue of the observable. Many observables have **quantized eigenvalues**, i.e. a measurement can only yield one of a discrete set of values. Right after the measurement, the state of the system is an eigenstate of the observable, which means that the value of the observable is exactly known.

- A state can be a simultaneous eigenstate of several observable, which means that the observer can exactly know the value of several properties of the system at the same time and make exact predictions about the outcome of measurements of those properties. But there are also incompatible observables whose exact values cannot be known to the observer at the same time.

- A state cannot be a simultaneous eigenstate of incompatible observables. If it is in an eigenstate of one of the incompatible observables and the value of this observable is known, then quantum mechanics gives only the probabilities for measuring each of the different eigenvalues of the other incompatible observables. The outcome of a measurement of any of the other incompatible observables is uncertain. A measurement of one of the other incompatible observables changes the state of the system to one of its eigenstates and destroys the information about the value of the first observable.

- To completely specify the initial state of a system with n degrees of freedom, we have to make up to n compatible measurements. To specify the state of this particle we have to make up to three compatible measurements.

- The possible outcome of the measurements are usually not specified directly, but through labels called **quantum numbers**.