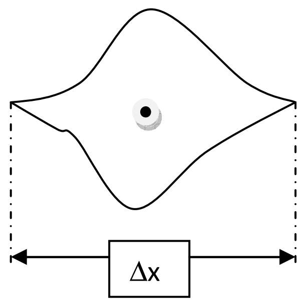
UNCERTAINTY PRINCIPLE

Introduction



In quantum mechanics a particle can be represented by a wave packet of a linear extension Δx (in one dimensional approach) surrounding the particle, which moves with the group velocity. Since the particle may be anywhere within the wave packet, it may be found anywhere within the limits Δx. Hence it can be said that **the position of the particle is uncertain by an amount Δx.**

Since a wave packet is formed by superposition of a large number of waves of different wavelength, hence the exact wavelength of a wave packet cannot be predicted with precision. It can only be said that the wavelength lies within a certain range λ to λ + dλ. The momentum of the particle being: $p = \frac{h}{\lambda}$, hence due to uncertainty in wavelength the momentum is also uncertain within a range p to p + dp, **dp being the momentum uncertainty**.

Now : $p = \frac{h}{\lambda}$. Hence $dp = -\frac{h}{\lambda^2}d\lambda$. Ignoring the negative sign the momentum uncertainty is given by:

$$\Delta p = \frac{h}{\lambda^2}\Delta\lambda. \quad \text{..(7.1)}$$

This shows that the momentum uncertainty is directly proportionate to the wavelength uncertainty. If the number of superposed waves in the wave packet and their range of wavelength be increased, then the linear spread Δx of the wave packet decreases leading to a reduction in position uncertainty. When $\Delta\lambda \rightarrow \infty$, $\Delta p \rightarrow \infty$ but $\Delta x \rightarrow 0$, i.e the wave packet reduces to a point and there is no uncertainty in the position of the particle.

On the other hand, for a definite λ, $\Delta\lambda = 0 \Rightarrow \Delta p = 0$. So there is no uncertainty in momentum of the particle. But in this case $\Delta x \rightarrow \infty$ and the uncertainty in position is limitless. **Thus it is evident that the uncertainty in position decreases at the expense of certainty of momentum and vice versa.** Heisenberg made an elaborate analysis of this uncertainty or indeterminacy and proposed the uncertainty principle.

Heisenberg uncertainty principle.

It is fundamentally impossible to determine the position and momentum of a particle simultaneously with absolute precision. If the uncertainty in one decreases themn that of the other will increase, the minimum value of the uncertainty product being 'h', the Planck's constant.

$$\Delta x.\Delta p \geq h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7.1)$$

Implication of uncertainty principle

At the macroscopic level, where we deal with particles of considerably large size and finite velocities, where any type of observation and measurement maintains the particle undisturbed. And the uncertainties can be ignored since 'h' is extremely small. However at the microscopic level, where atomic and sub-atomic particles are dealt with, any type of observation and measurement disturbs the particle. So an unavoidable indeterminacy will be associated with the measurements.

The uncertainty product $\Delta x.\Delta p$ is independent of the quantities like mass, wavelength etc. The minimum value of the uncertainty product 'h' is an universal constant. So the uncertainty principle is a fundamental law of nature.

The consequence of this inherent and unavoidable indeterminacy of observational results in micro world is that – only the concept of probability of getting a particular value for a physical quantity can be considered. The exact value cannot be predicted.

Other forms of uncertainty relation

The uncertainty of observation and measurement of a physical quantity is not limited only to position and momentum measurements. It can be extended to any two canonically conjugate quantities, for e.g. Energy and time, angular momentum and angular position etc.

1) Energy and time: For a particle of mass 'm' moving along X axis with momentum 'p' the energy is given by:

$$E = \frac{p_x^2}{2m} \Rightarrow \Delta E = \frac{p_x}{m}\Delta p_x = v_x \Delta p_x \quad \ldots\ldots\ldots\ldots\ldots(7.2)$$

Where v_x is the velocity of the particle in X direction.

$$V_x = \frac{\Delta x}{\Delta t} \text{ which gives } \Delta x.\Delta p_x = \boxed{\Delta E.\Delta t \geq h} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7.3)$$

Equation (7.3) is an alternative form of uncertainty principle.

2) Angular momentum and angular position

The energy E of a particle rotating about an axis with angular speed 'ω' moment of inertia is given by:

$E = \frac{1}{2}I\omega^2 \Rightarrow \Delta E = \frac{1}{2}.\,2\omega I.\,\Delta\omega = \omega.I.\Delta\omega$ ……………………………(7.4)

As I is independent of ω and depends only on the mass and dimension of the particle

∴ $I\Delta\omega = \Delta(I\omega) = \Delta L$, where $L = I\omega$ = angular momentum of the rotating particle.

This gives: $\Delta E = \omega\Delta L$ ………………………………………………(7.5)

The angular position with respect to an initial reference point is = $\phi = \omega t \Rightarrow \Delta\phi = \omega\Delta t$ …………(7.6)

From equation (7.5) and (7.6) : $\Delta L.\Delta\phi = \frac{\Delta E}{\omega}.\,\omega\Delta t = \Delta E.\Delta t$ ……………………………(7.7)

Substituting (7.3) in (7.7): $\boxed{\Delta L.\Delta\phi \geq h}$ ………………………(7.8)

Equation (7.8) is another alternative form of uncertainty principle.

<u>Some illustrations of uncertainty principle</u>

1. γ ray microscope thought experiment

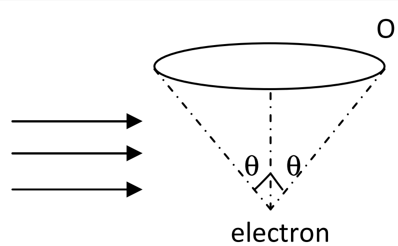


Fig. 7.1. Illustration of uncertainty relation – the γ ray microscope.

Let position and momentum determination of an electron be considered using an imaginary γ ray microscope of high resolving power (imaginary, as optical parts cannot focus γ rays) with objective O. The smallest distance resolved by the microscope is:

$\Delta x = \frac{\lambda}{2\mu \text{Sin}(\theta)}$, where μ is the refractive index of the concerned medium

and 2θ is the angle subtended by the objective lens at the position of the electron. λ is the wavelength of the γ radiation.

Refractive index for air is = 1, which gives:

$\Delta x = \frac{\lambda}{2\text{Sin}(\theta)}$ …………………………………(7.9)

Thus Δx is the minimum error in the measurement of electron. To reduce Δx, the wavelength λ should be reduced i.e. shorter γ rays are required.

In order to measure the position of the electron, at least one γ ray photon, deflected by the electron from its initial direction should enter the objective. This photon imparts a momentum of $\frac{h\nu}{c}$ to the electron, which thereby recoils. As the scattered photon can enter the objective from any direction within an angle $-\theta$ to θ, hence the momentum of the electron is uncertain to that extent. Thus X-component of momentum of the electron ranges from $-\frac{h\nu}{c}\mathrm{Sin}\theta$ to $+\frac{h\nu}{c}\mathrm{Sin}\theta$. Hence the minimum uncertainty in x component of momentum is

$$\Delta p_x = \frac{2h\nu}{c}\mathrm{Sin}\theta = \frac{2h}{\lambda}\mathrm{Sin}\theta \quad \text{...(7.10)}$$

From equations (7.9) and (7.10) the minimum value of the uncertainty product is.

$$\Delta x.\ \Delta p_x = \frac{\lambda}{2\mathrm{Sin}(\theta)} \cdot \frac{2h}{\lambda}\mathrm{Sin}\theta = h \quad \text{...(7.11)}$$

This verifies Heisenberg's uncertainty principle.

2. Diffraction of electron beam at a slit.

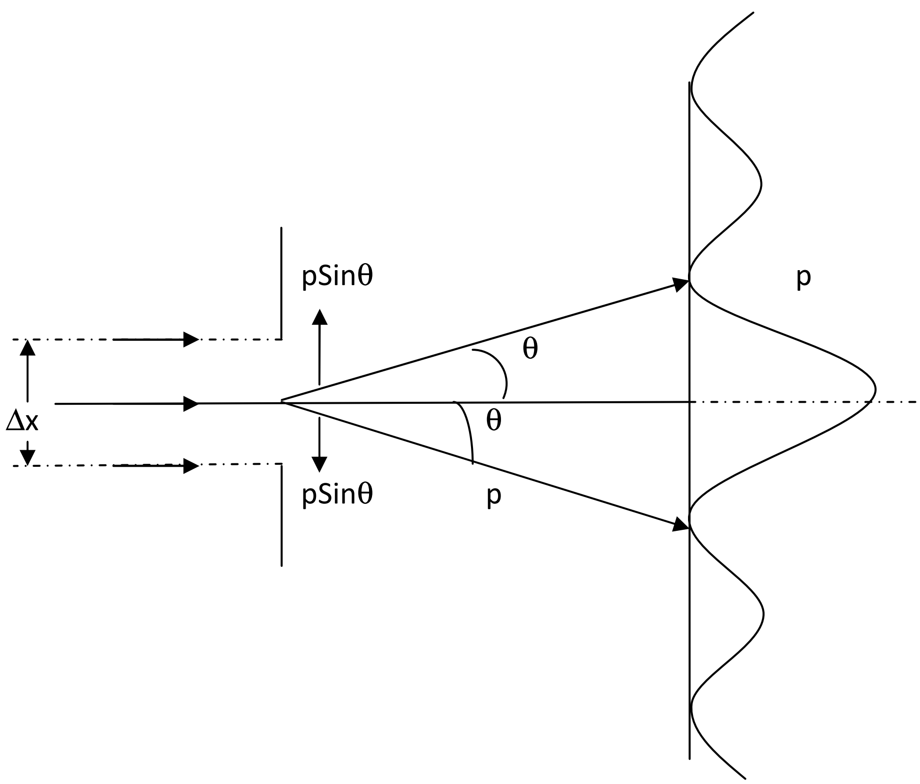


Fig. 7.2. Illustration of uncertainty relation – Single slit diffraction of electron beam.

The fig.(7.2) shows a single slit exposed to an electron beam. **The width of the slit Δx is the position uncertainty.** For single slit diffraction most of the intensity is concentrated within the central maxima. Let 'θ' be the angle of diffraction corresponding to first minima. Assuming that the secondary maximas tend to zero, almost all of the electrons will be scattered within an angle $-\theta$ to $+\theta$ on either side of direction of incidence. So if p be the momentum of the scattered electron then along a direction parallel to the width of the slit, p ranges from $-p\sin\theta$ to $+p\sin\theta$. **So the momentum uncertainty is $\Delta p_x = 2p\sin\theta$.** If λ be the de Broglie wavelength of the electron then: $P = \frac{h}{\lambda}$, which gives

$$\Delta p_x = \frac{2h}{\lambda}\sin\theta \quad \text{..(7.12)}$$

Now, according to Fraunhofer diffraction theory the condition for minima is :

$\Delta x\sin\theta = n\lambda$, where n = order of fringe.

For first minima n = 1, which gives

$$\Delta x\sin\theta = \lambda \quad \text{..(7.13)}$$

Or $$\Delta x = \frac{\lambda}{\sin\theta} \quad \text{...(7.14)}$$

From equation (7.14) and (7.12)

$$\boxed{\Delta x . \Delta p_x = \frac{\lambda}{\sin\theta} . \frac{2h}{\lambda}\sin\theta = 2h} \quad \text{..............................(7.15)}$$

As 2h is greater than 'h' it can be said that $\Delta x.\Delta p_x > h$, which is in agreement with uncertainty principle.