Open and Closed sets

Interior Point: Let $S \in \mathbb{R}$ and $a \in S$. Then a is called an interior point of S if $\exists\ \epsilon > 0$ $N_\epsilon(a) \subseteq S$.

Example 1: *Show that $\frac{1}{3}$ is an interior point of [0,1].*

Proof: Since, $\frac{1}{3} \in [0,1]$, and choose $\epsilon = \min\left\{\frac{1}{3} - 0, 1 - \frac{1}{3}\right\}$. Then, $\left(\frac{1}{3} - \epsilon, \frac{1}{3} + \epsilon\right) \subseteq [0,1]$. Thus, $\exists \epsilon > 0$, $N_\epsilon(\frac{1}{3}) \subseteq [0,1]$.

Therefore, $\frac{1}{3}$ is an interior point $[0,1]$. $\square$

Example 2: *Show that* 1 *is not an interior point of [0,1].*

Proof: If 1 is an interior point of $[0,1] \implies \exists \epsilon > 0$, $N_\epsilon(1) \subseteq [0,1]$.

Negation of the above statement is:

1 is not an interior point of $[0,1] \implies \forall \epsilon > 0$, $N_\epsilon(1) \not\subseteq [0,1]$.

Now, let $\epsilon > 0$ be arbitrary. Then $y = 1 + \frac{\epsilon}{2} \in N_\epsilon(1)$, however, $y \notin [0,1]$. Therefore, $N_\epsilon(1) \not\subseteq [0,1]$. As $\epsilon > 0$ is arbitrary, therefore, $\forall \epsilon > 0$, $N_\epsilon(1) \not\subseteq [0,1]$. Thus 1 is not an interior point of $[0,1]$. $\square$

Boundary Point: Let $S \in \mathbb{R}$ and $a \in \mathbb{R}$. Then a is called a boundary point of S if $\forall\ \epsilon > 0$ $N_\epsilon(a) \cap S \neq \Phi$ and $N_\epsilon(a) \cap S^c \neq \Phi$.

Example 3: *Show that* 1 *is a boundary point of* $[0,1)$.

Proof: We have to show, $\forall \epsilon > 0$, $N_\epsilon(1) \cap [0,1) \neq \Phi$, and $N_\epsilon(1) \cap [0,1)^c \neq \Phi$.

Take any $\epsilon > 0$, first shall show that, $N_\epsilon(1) \cap [0,1) \neq \Phi$ *i.e.* $\exists y \in N_\epsilon(1) \cap [0,1)$. Choose, $y = \max\left\{0, 1 - \frac{\epsilon}{2}\right\}$, then $y \in N_\epsilon(1)$ and $y \in [0,1)$.

Thus, $N_\epsilon(1) \cap [0,1) \neq \Phi$.

Now, we shall show that, $N_\epsilon(1) \cap [0,1)^c \neq \Phi$. Choose, $z = 1 + \frac{\epsilon}{2}$, then $z \in N_\epsilon(1)$ but $z \notin [0,1) \implies z \in [0,1)^c$.

Therefore, $N_\epsilon(1) \cap [0,1)^c \neq \Phi$.

Thus 1 is a boundary point of $[0,1)$. $\square$

Interior of a Set: The set $\mathbb{S}^o$ of all interior points of a set S is called the interior of S.

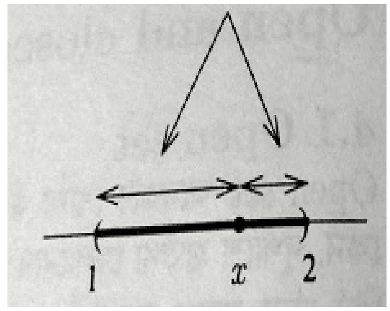
Open and Closed sets: Let $S \subseteq \mathbb{R}$.

1. S is said to be Open if $\forall x \in S\ \exists \epsilon > 0\ N_\epsilon(x) \subseteq S$.

2. S is said to be Closed if and only if $\mathbb{R} \setminus S$ is open.

Example 4: *Show that* $(1, 2)$ *is open in* $\mathbb{R}$.

Proof: To show $(1, 2)$ is open in $\mathbb{R}$ *i.e.* $\forall x \in (1, 2)\ \exists \epsilon > 0 \quad N_\epsilon(x) \subseteq (1, 2)$.



Let $x \in (1, 2)$ be any arbitrary point. Now, see the above figure, We choose $\epsilon = \min\{x - 1, 2 - x\} > 0$.

We shall now prove that $N_\epsilon(x) \subseteq (1, 2)$. Let, $y \in N_\epsilon(x)$. Then $y > x - \epsilon \geq x - x + 1 = 1$, as $\epsilon \leq x - 1$ and $\epsilon \leq 2 - x$.

Also, $y < x + \epsilon \leq x + 2 - x = 2$, as $\epsilon \leq x - 1$ and $\epsilon \leq 2 - x$.

Therefore, $y \in (1, 2) \implies N_\epsilon(x) \subseteq (1, 2)$.

Since, $x \in (1, 2)$, therefore, $\forall x \in (1, 2)\ \exists \epsilon > 0 \quad N_\epsilon(x) \subseteq (1, 2)$. $\square$

Example 5: Show that $\mathbb{S}^o$ is the largest open subset of $S \subseteq \mathbb{R}$.

Proof: Case 1 : If $\mathbb{S}^o = \Phi$, then the result is vacuously true.

Case 2 : If $\mathbb{S}^o \neq \Phi$, take any $a \in \mathbb{S}^o$. Therefore, a is an interior point of S *i.e.* $\exists \epsilon > 0$, $N_\epsilon(a) \subseteq S$.

We choose this ϵ. We have to show $N_\epsilon(a) \subseteq \mathbb{S}^o$.

Let, $x \in N_\epsilon(a)$, since $N_\epsilon(a)$ is open therefore, $\exists \delta > 0$ *s.t.* $N_\delta(x) \subseteq N_\epsilon(a) \subseteq S \implies x$ is an interior point of $S \implies x \in \mathbb{S}^o \implies N_\epsilon(a) \subseteq \mathbb{S}^o$.

Therefore, $\mathbb{S}^o$ is open.

Now, we shall show $\mathbb{S}^o$ is the largest open subset of S *i.e.* for any $A \subseteq S$ if A is open in $\mathbb{R}$, then we shall show $A \subseteq \mathbb{S}^o$. Take any $y \in A$, since A is open therefore, $\exists \epsilon > 0$, $N_\epsilon(y) \subseteq A \subseteq S \implies y \in \mathbb{S}^o$. Thus, $A \subseteq \mathbb{S}^o$. $\square$

Exercises

1. Prove that S is open $\iff$ $\mathbb{S}^o = S$.

2. Let $a \in \mathbb{R}$
   (a) Prove that $\{a\}$ is closed.
   (b) Using part (a), prove that every finite set must be closed.

3. Give an example of two sets A and B such that neither set is open but their union is open.

4. Let $S = [0, 1) \cap (1, 2)$. Then what is $\mathbb{S}^o$?

5. Show that the set

   $S = \{x \in \mathbb{R} : |x - 1| + |x - 2| < 3\}$ is an open set.

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