Real analysis (Sem 2)

T.S.

Terminating decimals: It's a decimal that has finite number.

Example $\frac{1}{2} = 0.\underbrace{5}_{\text{1 decimal digit}}$, $0.\underbrace{25}_{\text{2 decimal digit}}$

Non-terminating decimal: It has digits that go on forever. Example $\frac{1}{3} = 0.333\ldots$

Expressing a terminating decimals into a non-terminating decimals: Example $\frac{1}{2} = 0.5$
$\simeq 0.499999\ldots$

Thus every terminating decimal can be uniquely expressed as a non-terminating decimals.

Proposition. ~~Conclusion~~: Every real number has unique non-terminating decimal expansion, and every non-terminating decimal expansion specifies a unique real number.

Problem: Show that $(0,1)$ is not a denumerable set.

Sol^n: Solution is by contradiction:

Let if be possible that $(0,1)$ is denumerable.

Then there is a bijection

$$f : \mathbb{N} \to (0,1)$$

Since, $f(1) \in (0,1)$, $f(2) \in (0,1)$, ... $f(n) \in (0,1)$, ...

Thus each $f(n)$ has a unique non-terminating decimal expansion. i.e

$$f(1) = 0.d_{11}\ d_{12}\ d_{13} \cdots$$
$$f(2) = 0.d_{21}\ d_{22}\ d_{23} \cdots$$
$$f(3) = 0.d_{31}\ d_{32}\ d_{33} \cdots$$
$$\vdots$$
$$f(n) = 0.d_{n1}\ d_{n2}\ d_{n3} \cdots$$
$$\vdots$$

Now consider the real number

$$\alpha = 0.\alpha_{11}\ \alpha_{22}\ \alpha_{33} \cdots \quad (*)$$

where $\alpha_{nn} = \begin{cases} 2 & \text{if } d_{nn} = 1 \\ 1 & \text{otherwise.} \end{cases}$ then $\alpha_{nn} \neq d_{nn}$ for all $n \in \mathbb{N}$

clearly $x \in (0,1)$ with unique non-terminating expansion $(*)$

However there exist no $n \in \mathbb{N}$ s.t $f(n) = x$.

Thus f is not a bijection.

Thus $(0,1)$ can not be denumerable.

Exersise 2. Show that $[0,1]$ is not denumerable

Sol^n: Same Solution as exersise 1

Exersise 3: If $a < b$ then show that the interval $[a,b]$ is uncountable.

Sol^n: → Step 1 First show $[0,1]$ is uncountable

Step 2 Consider the map ~~$f: [0,1] \to [a,b]$~~

~~$f(x) = a + (b-a)x$~~ $f: [0,1] \to [a,b]$

$$f(x) = a + (b-a)x.$$

Show that f is a bijection.

Thus $[0,1]$ and $[a,b]$ are equipotent.

Since $[0,1]$ is uncountable therefore $[a,b]$ is uncountable.

Exersise 4 Show that $\mathbb{R}$ is uncountable.

Exersise 5 Show that there can not be a 1-1 map $f: [0,1] \to \mathbb{N}$

Exersise 6 Show that there can not be a 1-1 map $f : \mathbb{R} \to \mathbb{N}$

Exersise 7 Is there possible to have a one to one map $f : \mathbb{N} \to [0,1]$?

Exersise 8 Provide a bijection from $f : [0,1] \to [0,1)$

— — — — end of the chapter on Countability — — —