Cyclic group

Definition: A group G is called cyclic if there is an element a in G such that

$G = \{a^n | n \in \mathbb{Z}\}$. Such an element a is called a generator of$G$.

we may indicate that G is a cyclic group generated by a by writing $G = < a >$.

In additive notation, $G = \{na | n \in \mathbb{Z}\} = < a >$.

Observation:

1. $a$must have an order.
2. We know that if $\circ(a) = n$ then $\{a, a^2, \ldots, a^{n-1}, a^n(= e)\}$ are all distinct elements.
   Hence G can be finite/infinite.
   $\circ(a)$is finite iff G is finite
   $\circ(a)$is infinite iff$G$ is infinite
3. $|G| = | < a > | = \circ(a)$

Questions:

4. Do you think$G = < a^{-1} >$? If it is true can we say a^{-1} is also a generator?
5. Do you think cyclic implies abelian here?
6. What about the countabilityof G? [hint: Z is countable]

Example:

1. $(Z, +)$is a cyclic group as $Z = < 1 >$.
2. $(Z_n, +)$is a cyclic group as $Z_n = < \bar{1} >$.

Exercise : [hint: search for an element a and verify $G = < a >$.]

1. $Z_8$is a cyclic group.
2. $U(10)$is a cyclic goup.

Theorem: Let $< a >$ be a finite cyclic group. Then $| < a > | = \circ(a)$.

Proof: see Theorem 2.12.3 in S.K.Mapa

Theorem: Let $< a >$ be an infinite cyclic group. Then $| < a > | = \circ(a)$.

Proof: left to the reader.

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Theorem: Let $G = < a >$ be a finite cyclic group of order n. Then for a positive integer k, a^k is also a generator of G iff $\gcd(k, n) = 1$.

Proof: Theorem 2.12.6 in S.K.Mapa

Corollary: Total number of generator of a finite cyclic group G of order n is $\phi(n)$.

Exercise: Find all generatorsof Z_{10}.

Theorem: Every subgroup of a cyclic group is cyclic.

Proof: see any book.

Theorem: A cyclic group of finite order n has one and only one subgroup of order d for every positive divisor $d$of n.

Proof: see any book.

Exercise:

1. Find all subgroups of $(Z, +)$. [worked out in S.K.Mapa p114]
2. Prove that $(Q, +)$ is non cyclic. Deduce that $(R, +)$ is non cyclic.
3. Prove that $(Q^*, .)$ is non cyclic.
4. Let G be a group such that $|G| = mn,\ m > 1, n > 1$. Show that G has a non trivial subgroup.
5. Let $G = < a >$ be a cyclic group of order 30. Determine$< a^5 >, < a^2 >$.
6. A cyclic group of prime order has no proper non trivial subgroup.
7. If an abelian group G contains an element of order 5, prove that G must be a cyclic group.