Syllabus:

1. Solve first order differential equation by Euler's method
2. Solve first order differential equation by Improved Euler's method
3. Solve first order coupled equations by Euler's method
4. Solve second order coupled equations by Euler's method

Euler's method

Euler's method numerically approximates solutions of first-order ordinary differential equations(ODEs) with a given initial value.

$$\frac{dy(t)}{dt} = f\big(t, y(t)\big)$$

initial value: $y(t_0) = y_0$

To get a numeric solution, we replace the derivative on the LHS with a finite difference approximation

$\frac{dy(t)}{dt} \approx \frac{y(t+h)-y(t)}{h}$ h is taken small for better result

then $$y(t+h) \approx y(t) + h\frac{dy(t)}{dt}$$

or, $$y(t+h) \approx y(t) + hf(t, y(t))$$

The iterative solution rule is then:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

where h is called step size, the most relevant parameter for accuracy of the solution.

Example

$\frac{dy}{dx} = x + y + xy, initial\ cond: x_0 = 0, y_0 = 1 \rightarrow y(0) = 1$Use Euler's Method with a step size of h=0.025 to find approximate values of the solution at y = 0.05, 0.075, 0.1, 0.4 and Compare them to the exact values of the solution at these points.

Algorithm:

Define function : $f(x, y) = x + y + xy$

Initial condition :$x_0 = 0, y_0 = 1$

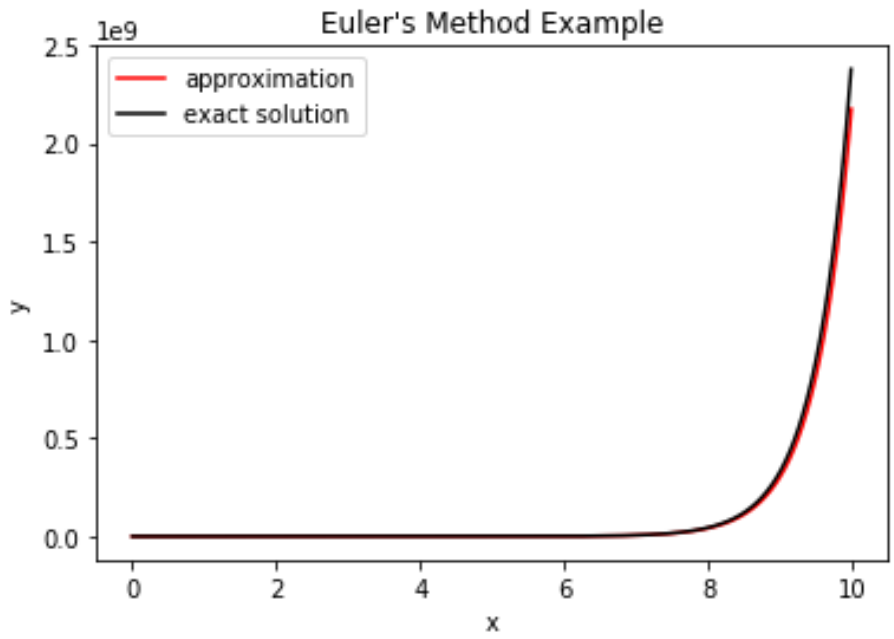
Step 1 : $h = 0.025$

Initial cond/values		$f(x_0, y_0) = x_0 + y_0 + x_0 y_0$	$y_{(x0+h)} = y_0 + fh$
x_0	y_0		
0	1	0+1+0=1	1+1*.025=1.025
0.025	1.025	0.025+1.025+1.025*.025= 1.075	1.025+1.075*.025=1.051
0.050	1.051	.050+1.051+.050*1.051=1.153	1.051+1.153*.025=1.079
0.075	1.079	.075+1.079+.075*1.079=1.234	1.079+1.234*.025=1.109
0.1	1.109		

Ex:1 / Euler1

dy / dx =2y calculate y with initial cond at x=0, y=5

```
import numpy as np
import matplotlib.pyplot as plt
deff(pr,x,y ):
    #parameter set None from main
    return 2*y
x = 0.0
y = 5.0
dx=0.005
X = 10
# Function for euler formula
#defeuler(f, pr, x, y, h, X ):
xx,yy,ddydx=[],[],[]
while abs(x) < abs(X):
dydx=f(None,x,y)
xx.append(x); yy.append(y); ddydx.append(dydx)
    y+=dydx*dx
    x+=dx

#plot
plt.plot(xx,yy,'r', label='approximation' )
x1=np.arange(0,10,0.01)
plt.plot(x1,5.0*np.exp(2*x1),'k-',label='exact solution')
plt.title( "Euler's Method Example" )
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.show()

# Printing approximation
print('h = '"%.3f"% h)
print('Approximate solution at X=10  is'" %f" %y)
```



h = 0.005

Approximate solution at X=10 is 2196431025.250482

Improved Euler method:

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))]$$

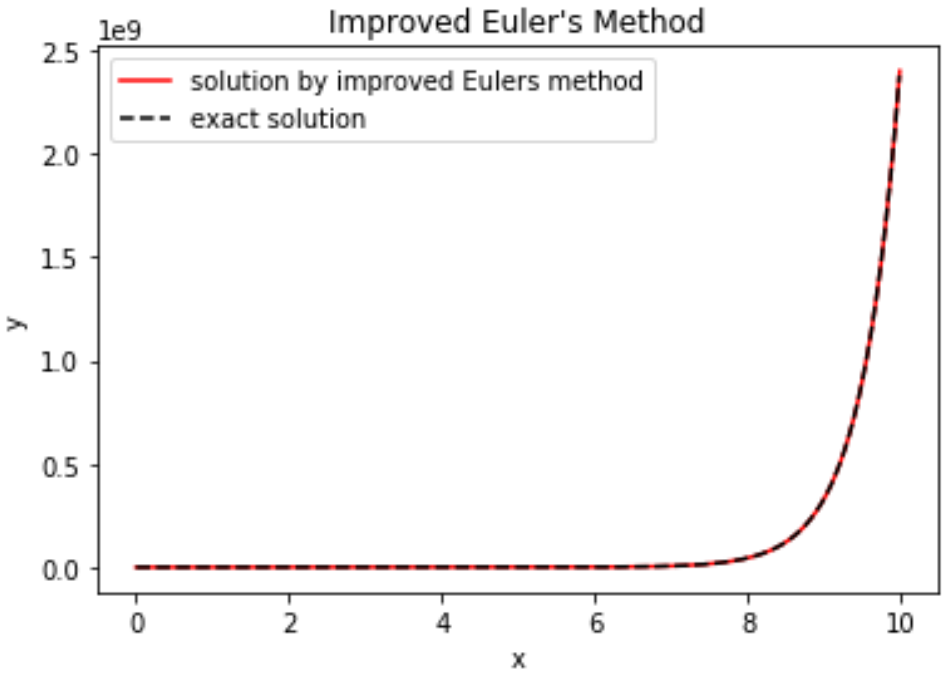
Ex:2 / ImEuler1

dy / dx =2y calculate y with initial cond at x=0, y=5

#**improved Euler method**

```
import numpy as np
import matplotlib.pyplot as plt
deff(pr,x,y ):
    #parameter set None from main
    return 2*y
x = 0.0
y = 5.0
dx=0.005
X = 10
xx,yy,ddydx=[],[],[]
while abs(x) < abs(X):
```

```
dydx=0.5*(f(None,x,y)+f(None,x+dx,y+dx*f(None,x,y)))
xx.append(x); yy.append(y); ddydx.append(dydx)
    y+=dydx*dx
    x+=dx

#plot
plt.plot(xx,yy,'r', label='solution by improved Eulers method' )
x1=np.arange(0.0,10.0,0.01)
plt.plot(x1,5.0*np.exp(2*x1),'k--',label='exact solution')
plt.title( "Improved Euler's Method" )
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.show()

 # Printing approximation
print('h = '"%.3f"% dx)
print('Approximate solution at X=10  is'" %f" %y)
```




```
h = 0.005
Approximate solution at X=10  is 2425023541.445030
```

Ex:3 /EULER2

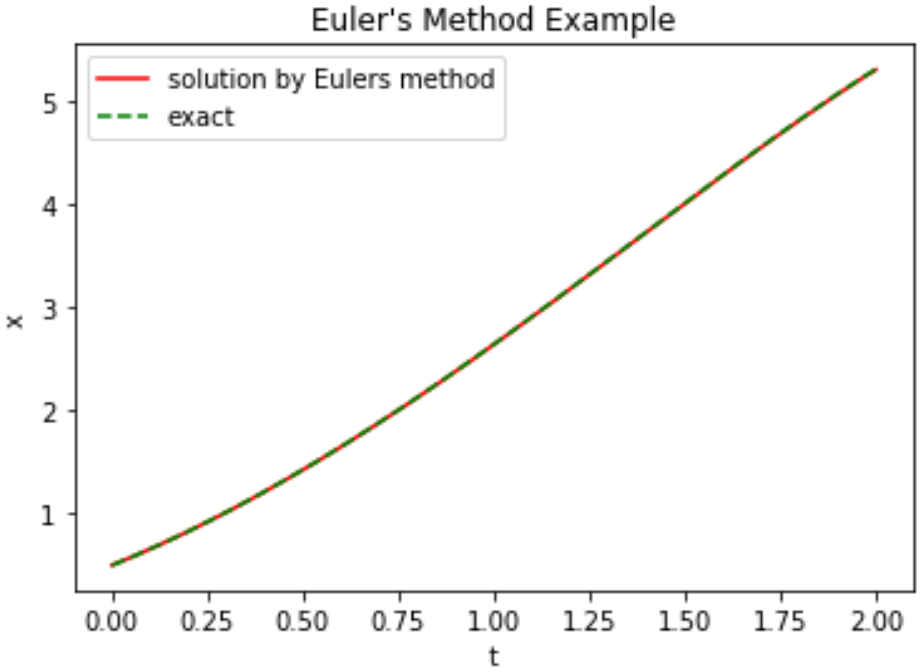
dy / dx =x-t^2+1 , 0<=t<=2, calculate y with initial cond at x=0.5, t=0

#For reference, the analytical solution is x=(t+1)^2-0.5exp(t)

#calculate error=$\varepsilon = \sum_1^N |x(t_i) - x_i|$

```
import numpy as np
import matplotlib.pyplot as plt
deff(pr,t,x ):
    #parameter set None from main
    return x-t**2+1
t = 0.0
x = 0.5
h=(2.0-0.0)/1000.0
T = 2.0
tt,xx,ddxdt=[],[],[]
while abs(t) < abs(T):
dxdt=f(None,t,x)
tt.append(t); xx.append(x); ddxdt.append(dxdt)
    x+=dxdt*h
    t+=h

#plot
plt.plot(tt,xx,'r', label='solution by Eulers method' )
t1=np.arange(0.0,2.0,0.001)
plt.plot(t1,(t1+1.0)**2-0.5*np.exp(t1),'g--',label='exact')
plt.title( "Euler's Method Example" )
plt.xlabel('t')
plt.ylabel('x')
plt.legend()
plt.show()

 # Printing approximation
print('h = '"%.3f"% h)
```

```
print('Approximate solution at X=2  is''' %f" %x)
```



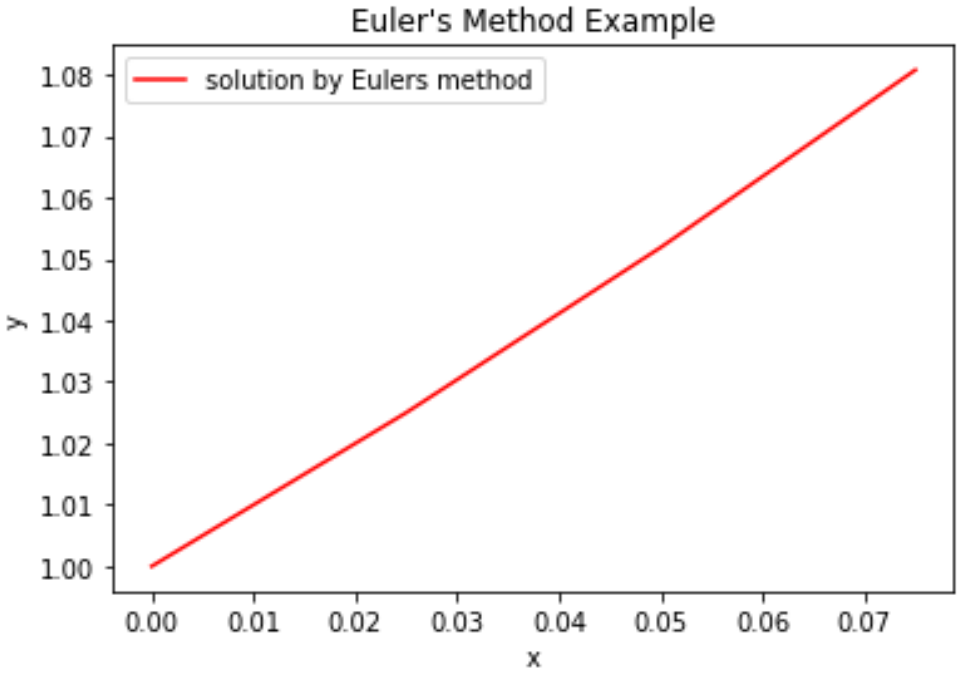
h = 0.002
Approximate solution at X=2 is 5.300095

Ex:4 / Euler3

#dy / dx =x+y+xy calculate y at x=0.1 with initial cond at x=0, y=1

```
import numpy as np
import matplotlib.pyplot as plt

deff(pr,x,y ):
    #parameter set None from main
    return x+y+x*y
x = 0.0
y = 1.0

h=0.025
X = 0.1


xx,yy,ddydx=[],[],[]
```

```
while abs(x) < abs(X):
dydx=f(None,x,y)
xx.append(x); yy.append(y); ddydx.append(dydx)
    y+=dydx*h
    x+=h
#plot
plt.plot(xx,yy,'r', label='solution by Eulers method' )
#x1=np.arange(0.0,1.0,0.01)
#plt.plot(x1,,'k-',label='exact solution')
plt.title( "Euler's Method Example" )
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.show()
 # Printing approximation
print('h = '"%.3f"% h)
print('Approximate solution at X=0.1  is'" %f" %y)
```



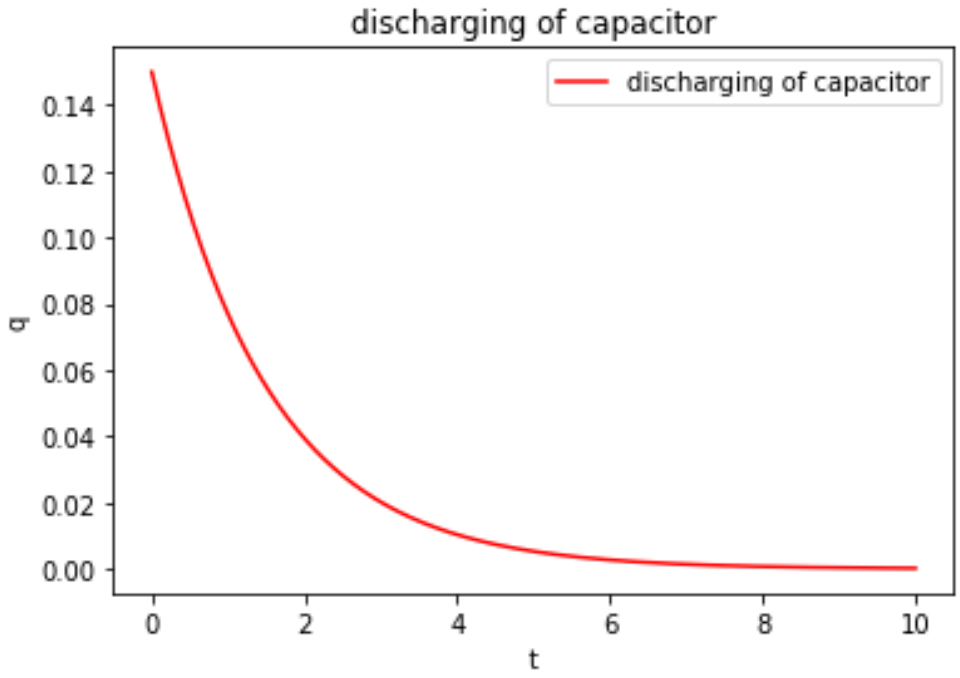

```
h = 0.025
Approximate solution at X=0.1  is 1.111673
```

Ex :5 / Euler4

#dq / dt =-q/RC calculate q at t=10 with initial cond at t=0, q=CE
discharging of capacitor

```
import numpy as np
import matplotlib.pyplot as plt

deff(RC,t,q ):
    return -q/(R*C)

R,C =100,1.5e-2
RC=R*C
E= 10.0

t = 0.0
q = E*C

h=0.01
T = 10

tt,qq,ddqdt=[],[],[]
while abs(t) < abs(T):
dqdt=f(RC,t,q)
tt.append(t); qq.append(q); ddqdt.append(dqdt)
    q+=dqdt*h
    t+=h

#plot
plt.plot(tt,qq,'r', label='discharging of capacitor' )
plt.title( "discharging of capacitor" )
plt.xlabel('t')
plt.ylabel('q')
plt.legend()
plt.show()

 # Printing approximation
print('h = '"%.3f"% h)
```

```
print('Approximate solution at T=10  is'" %f" %q)
```



h = 0.010

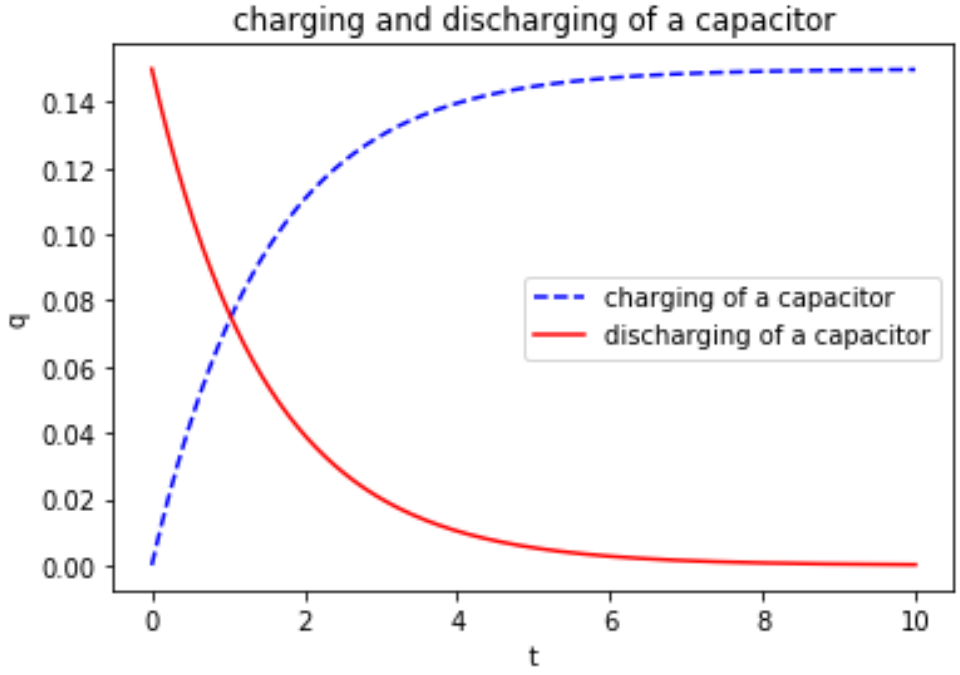
Approximate solution at T=10 is 0.000185

Ex :6/Euler5

```
#dq / dt =-q/RC calculate q at t=10 with initial cond at t=0, q=CE/discharging a capacitor
#dq / dt =E/R-q/RC calculate q at t=10 with initial cond at t=0, q=CE/charging a capacitor
import numpy as np
import matplotlib.pyplot as plt

deff(RC,t,q ):
    return -q/(R*C)

def f2(RC,t,q ):
    return (E/R)-q/(R*C)

R,C =100,1.5e-2
E= 10.0
RC=R*C
t = 0.0
```

```
q = 0.0

#for charging
h1=0.01
T1 = 10

tt,qq,ddqdt=[],[],[]
while abs(t) < abs(T1):
dqdt=f2(RC,t,q)
tt.append(t); qq.append(q); ddqdt.append(dqdt)
   q+=dqdt*h1
   t+=h1
plt.plot(tt,qq,'b--', label='charging of a capacitor' )
print('h1 = '"%.3f"% h1)
print('Approximate value of charge at T=10 during CHARGING  is'" %f" %q)

#for discharging
h=0.01
T = 10

t = 0.0
q = E*C

tt,qq,ddqdt=[],[],[]
while abs(t) < abs(T):
dqdt=f(RC,t,q)
tt.append(t); qq.append(q); ddqdt.append(dqdt)
   q+=dqdt*h
   t+=h
plt.plot(tt,qq,'r', label='discharging of a capacitor' )
print('h = '"%.3f"% h)
print('Approximate value of charge at T=10 during DISCHARGING   is'" %f" %q)

#plot
plt.title( "charging and discharging of a capacitor" )
plt.xlabel('t')
```

```
plt.ylabel('q')
plt.legend()
plt.show()
```



h1 = 0.010

Approximate value of charge at T=10 during CHARGING is 0.149815

h = 0.010

Approximate value of charge at T=10 during DISCHARGING is 0.**000185**

Euler Method for coupled First order differential equations

$$\frac{dx}{dt} = f1(t, x, y)$$

$$\frac{dy}{dt} = f2(t, x, y)$$

x, y : dependent variable, t: independent variable.

Solve for x and y , given at t=0, $x=x_0$ and $y=y_0$?

Choose h.

Start from initial cond t=0 and use these eqs to get final result.

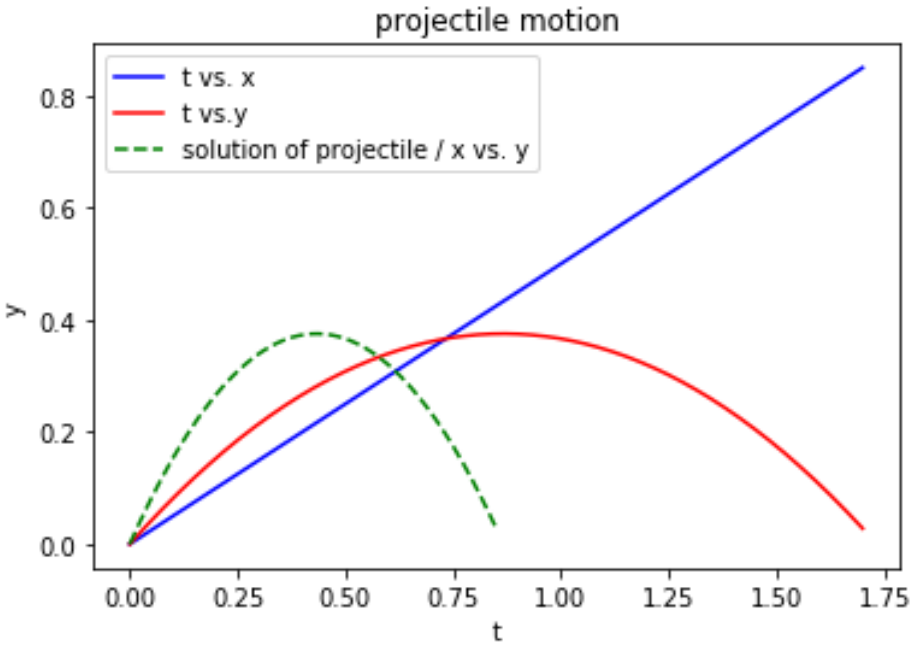
$$x(t+h) = x(t) + hf1(t,x,y)$$

$$y(t+h) = y(t) + hf2(t,x,y)$$

Ex:7 / Eulerprojectile

```
#A particle is projected with initial velocity u making an angle theta= pi/3
# with the horizontal.dx/dt=u cos(theta), dy/dt= u sin(theta)-gt,
# initial condition: at t=0, x=0,y=0. Take u=1, g=1
import numpy as np
import matplotlib.pyplot as plt
def f1(ux,t,x ):
    return ux

def f2(uy,t,y ):
    return uy-g*t

u =1.0
th = np.pi/3.0
ux = u*np.cos(th)
uy = u*np.sin(th)

t = 0.0
x = 0.0
y = 0.0
g = 1.0

dt=0.001
T = 1.7
```

```
xx,tt,ddxdt=[],[],[]
while abs(t) < abs(T):
dxdt=f1(ux,t,x)
tt.append(t); xx.append(x); ddxdt.append(dxdt)
    x+=dxdt*dt
    t+=dt
#print('*******************************')

t = 0.0

yy,tt,ddydt=[],[],[]
while abs(t) < abs(T):
dydt=f2(uy,t,y)
tt.append(t); yy.append(y); ddydt.append(dydt)
    y+=dydt*dt
    t+=dt

#plot
plt.plot(tt,xx,'b', label='t vs. x' )
plt.plot(tt,yy,'r', label='t vs.y' )
plt.plot(xx,yy,'g--', label='solution of projectile / x vs. y')
plt.title( "projectile motion" )
plt.xlabel('t')
plt.ylabel('y')
plt.legend()
plt.show()
```

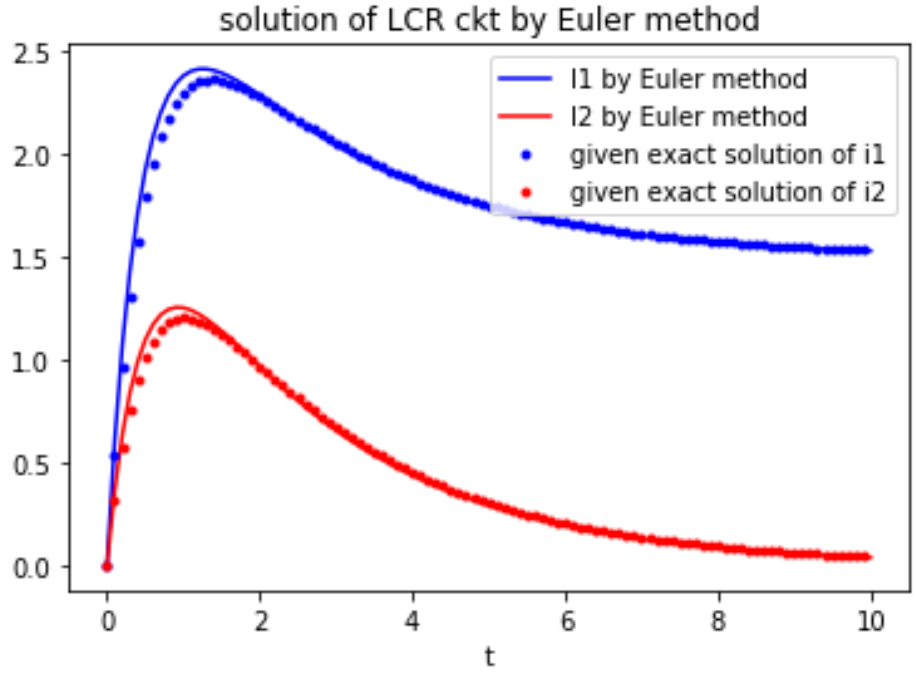


EX:8 /LCR

```
# LCR circuit   dI1/dt=6-4I1+3I2 and dI2/dt=3.6-2.4I1+1.6I2
#initial conditions at t=0, I1=I2=0
#exact solutions: I1=-3.375exp(-2t)+1.875exp(-0.4t)+1.5
#I2=-2.25exp(-2t)+2.25exp(-0.4t)
import numpy as np
import matplotlib.pyplot as plt

def f1(pr1,t,i1,i2):
    return 6.0-4.0*i1+3.0*i2

def f2(pr2,t,i1,i2):
    return 3.6-2.4*i1+1.6*i2

t=0.0
i1=0.0
i2=0.0
```

```
dt=0.1
T=10.0

tt = []
ii1, ddi1dt =[],[]
ii2, ddi2dt =[],[]

while abs(t)<abs(T):
    di1dt = f1(None,t,i1,i2)
tt.append(t)
    ii1.append(i1); ddi1dt.append(di1dt)
    di2dt = f2(None,t,i1,i2)
    ii2.append(i2); ddi2dt.append(di2dt)
    i1+=di1dt*dt
    i2+=di2dt*dt
    t+=dt
    print("%.2f"%i1,"%.2f"%i2,"%.1f"%t)

plt.plot(tt,ii1,'b',label='I1 by Euler method')
plt.plot(tt,ii2,'r',label='I2 by Euler method')
plt.title('solution of LCR ckt by Euler method')
plt.xlabel('t')
plt.legend()

tt=np.arange(0.0,10,0.1)
i1=-3.375*np.exp(-2*tt)+1.875*np.exp(-0.4*tt)+1.5
i2=-2.25*np.exp(-2*tt)+2.25*np.exp(-0.4*tt)
plt.plot(tt,i1,'b.',label='given exact solution of i1')
plt.plot(tt,i2,'r.',label='given exact solution of i2')
plt.legend()
```

plt.show()



HW.

1. Solve for half wave rectifier and full wave rectifier
2. Error

Solve 2nd order differential equation by Euler method

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$$

First split it into 2 equations.

$\frac{dy'}{dx} = f(x, y, y')$ and $\frac{dy}{dx} = y'$ and then solve y' and then for y.

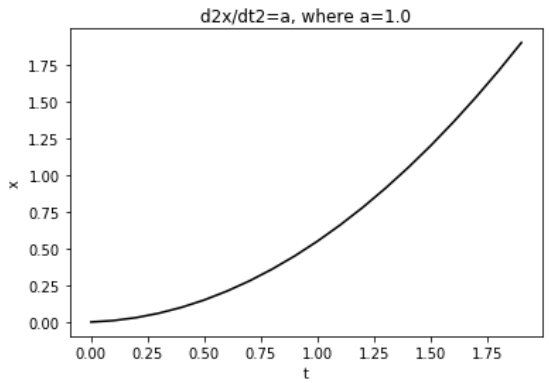
d2x/dt2=a, Newton's law of motion

import numpy as np

import matplotlib.pyplot as plt

Ex : 9 / newton

```
def f(a,t,x,dxdt):
    return a

a=1.0

t,x,dxdt=0.0,0.0,0.0

dt=0.1
T=2.0

tt, xx,  ddxdt, dd2xdt2=[], [], [], []
while abs(t)< abs(T):
    d2xdt2 = f(a,t,x,dxdt)
tt.append(t); xx.append(x); ddxdt.append(dxdt); dd2xdt2.append(d2xdt2)
dxdt+=d2xdt2*dt
    x+=dxdt*dt
    t+=dt
plt.plot(tt,xx, 'k',)
plt.title('d2x/dt2=a, where a=1.0')
plt.xlabel('t')
plt.ylabel('x')
plt.show()
```



Ex:10 / Forcedvibration