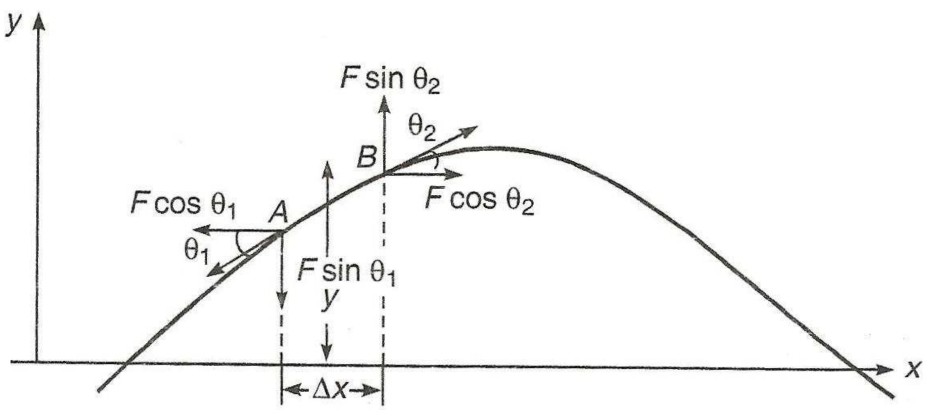
Waves on a StretchedString

Consider a uniform stretched string, having mass per unit length m. Under equilibrium conditions, it can be considered to be straight. The x-axis is chosen along the length of the stretched string in its equilibrium state. Let the string be displaced perpendicular to its length by a small amount so that a small section of length Δx is displaced through a distance y from its mean position, as shown in below. When the string is released, it results in wave motion.



We have studied that the wave disturbance travels from one particle to another due to their masses (or inertia) and the factor responsible for the periodic motion of the particle is the elasticity of the medium. For a stretched string, the elasticity is measured by the tension F in it and the inertia is measured by mass per unit length or linear mass density,m.

Suppose that the tangential force on each end of a small element AB is F. The force on the end B is produced by the pull of the string to the right and at A is due to the pull of the string to the left. Due to the curvature of the element AB, the forces are not directly opposite to each other. Instead, they make angles θ_1 and θ_2 with the x-axis. This means that the forces pulling the element AB at opposite ends, though of equal magnitude, do not exactly cancel each other. In order to calculate the net force along the x- and y-axes, the forces are resolved into rectangular components. The net force in the x and the y directions are respectively givenby

$$F_x = F\,cos\theta_2 - Fcos\,\theta_1$$

and

$$F_y = F\sin\theta_2 - Fsin\,\theta_1$$

For small angle approximation, $cos\theta \approx 1$ and$sin\theta \approx \theta \approx tan\theta.$ This implies that if the displacementofthestringperpendiculartoitslengthisrelativelysmall,theangles $\theta_1$and$\theta_2$will be small and there is no net force in the x-direction, and the element AB is only subjected to a net upward force F_y. Under the action of this force, the string element will move up and down. Therefore,they-componentoftheforceonelementABcanbewrittenas

$$F_y = F\tan\theta_2 - F\,tan\,\theta_1$$

We know that the tangent of an angle actually defines the slope at that point. In other words, the tangent define the derivative dy/dx. Using this result, the y-component of force on the element can be approximatedas

$$F_y = F\left(\frac{dy}{dx}\bigg|_{x+\Delta x} - \frac{dy}{dx}\bigg|_{x}\right)$$

Note that the perpendicular displacement y(x, t) of the string is both a function of the position x and time t. However, above equation is valid at a particular instant of time. Therefore, the derivative in this expression should be taken by keeping the time fixed. Therefore, equation can be rewrittenas

$$F_y = F\left(\frac{\partial y}{\partial x}\bigg|_{x+\Delta x} - \frac{\partial y}{\partial x}\bigg|_{x}\right)$$

For the sake of convenience, let us put

$$f(x) = \frac{\partial y}{\partial x}\bigg|_x \text{ and } f(x+\Delta x) = \frac{\partial y}{\partial x}\bigg|_{x+\Delta x}$$

$$F_y = F[f(x+\Delta x) - f(x)]$$

Therefore

Lets use Taylor series expansion of $f(x+\Delta x)$ about x

$$F_y = F[f(x) + \frac{\partial f}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Delta x^2 - f(x)]$$

$$= F[\frac{\partial f}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Delta x^2]$$

Since, Δx is small , we can ignore the second and the higher order terms in Δx

$$F_y = F\frac{\partial f}{\partial x}\Delta x = F\frac{\partial^2 y}{\partial x^2}\Delta x$$

ThisequationgivesthenetforceontheelementAB.WeuseNewton'ssecondlawofm otionto obtaintheequationofmotionofthiselement,byequatingthisforcetotheproductofm assand accelerationoftheelementAB.ThemassoftheelementABis$m\Delta x$.Therefore,we canwrite

$$m\Delta x\ \frac{\partial^2 y}{\partial t^2} = F\frac{\partial^2 y}{\partial x^2}\Delta x$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{m}{F}\frac{\partial^2 y}{\partial t^2}$$

Note that this equation has been obtained for a small element AB, but it can be applied to the entire string, since there is nothing special about this particular element of the string. In other words, this equation can be applied to all the elements of thestring.

Now, we know that the sinusoidal wave propagating on the string described by the equation

$$y(x,t) = Asin(\omega t - kx)$$

Therefore, $\frac{\partial^2 y}{\partial x^2} = -k^2 A \sin(\omega t - kx)$ and $\frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(\omega t - kx)$

Substituting in above equation , we get

$$-k^2 A\sin(\omega t - kx) = \frac{m}{F}[-\omega^2 A\sin(\omega t - kx)]$$

Or, $\frac{F}{m} = \left(\frac{\omega}{k}\right)^2$

But,weknowthatω/kisthewavespeedv,therefore,weget

$$v = \frac{\omega}{k} = \sqrt{\frac{F}{m}}$$

The above relation tells us that velocity of a transverse wave on a stretched string

depends on tension and mass per unit length of the string.

So, we can write 2

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Or, $$v^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

This is one-dimensional wave equation. It holds as long as we deal with small amplitude waves. Elasticity provides the restoring force and the inertia determines the response of themedium.

Q1.The wave speed of a wave on a string depends on the tension and the linear mass density. If the tension is doubled, what happens to the speed of the waves on the string?

Answer: The wave speed would increase by √2.

Q2. If the tension in a string were increased by a factor of four, by what factor would the wave speed of a wave on the string increase?

The wave speed is proportional to the square root of the tension, so the speed is doubled.

Q3.Guitars have strings of different linear mass density. If the lowest density string and the highest density string are under the same tension, which string would support waves with the higher wave speed?

Since the speed of a wave on a string is inversely proportional to the square root of the linear mass density, the speed would be higher in the low linear mass density of the string.

Q4.Two strings, one with a low mass density and one with a high linear density are spliced together. The higher density end is tied to a lab post and a student holds the free end of the low-mass density string. The student gives the string a flip and sends a pulse down the strings. If the tension is the same in both strings, does the pulse travel at the same wave velocity in both strings? If not, where does it travel faster, in the low density string or the high density string?

Try……

Q5.A sinusoidal wave travels down a taut, horizontal string with a linear mass density of μ=0.060kg/m. The maximum vertical speed of the wave is vymax=0.30cm/s. The wave is modeled with the wave equation y(x,t)=Asin(6.00m−1x−24.00s−1t). (a) What is the amplitude of the wave? (b) What is the tension in the string?

(a) A=0.0125cm; (b) F=0.96N

Q6. The speed of a transverse wave on a string is v=60.00m/s and the tension in the string is FT=100.00N. What must the tension be to increase the speed of the wave to v=120.00m/s?

Lets solve $v^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$**by method of separation of variables**

Take a trial solution, $y(x,t) = X(x)T(t)$

$\frac{\partial y}{\partial x} = \left(\frac{dX}{dt}\right)Tand\frac{\partial^2 y}{\partial x^2} = \left(\frac{d^2X}{dt^2}\right)T$ and $\frac{\partial^2 y}{\partial t^2} = \left(\frac{d^2T}{dt^2}\right)X$

Substituting, we get $v^2\left(\frac{d^2X}{dt^2}\right)T = \left(\frac{d^2T}{dt^2}\right)X$

Dividing by XT, $\frac{v^2}{X}\left(\frac{d^2X}{dt^2}\right) = \frac{1}{T}\left(\frac{d^2T}{dt^2}\right) = -\omega^2$

This is because, LHS depends on x only, and RHS depends on t only and x, t are independent variables. So both LHS and RHS qual to a constant, $-\omega^2(say)$.

There will be 2 equations: $\frac{v^2}{X}\left(\frac{d^2X}{dt^2}\right) = -\omega^2 \rightarrow \left(\frac{d^2X}{dt^2}\right) + \frac{\omega^2}{v^2}X = 0$

and

$\frac{1}{T}\left(\frac{d^2T}{dt^2}\right) = -\omega^2 \rightarrow \left(\frac{d^2T}{dt^2}\right) + \omega^2 = 0$

Solutions are: $X = A\cos\left(\frac{\omega}{v}\right)x + B\sin\left(\frac{\omega}{v}\right)x = Acos\ kx + Bsin\ kx$,

k=w/c

and $T = Ccos\ \omega t + D\sin\omega t$

A,B,C,D are constants to be determine for specific problem

Solution: $y = (Acos\ kx + Bsin\ kx)(Ccos\ \omega t + D\sin\omega t)$

Therefore,**the equation of wave passing through stretched string is**

$$\boldsymbol{y = (Acos\ kx + Bsin\ kx)(Ccos\ \omega t + D\sin\omega t)}$$

Let String fixed at both ends

Let the end points are x=0 and L

Therefore, initial conditions: y=0 at x=0, for all t

and y=0 at x=L for all t

Substituting, we get

For x=0, $0 = (A + 0)(C\cos\omega t + D\sin\omega t)$, this gives A=0

Therefore, : $y = B\sin kx(C\cos\omega t + D\sin\omega t)$

For x=L, $0 = B\sin kL(C\cos\omega t + D\sin\omega t)$,

This gives, $kL = n\pi \rightarrow k_n = \frac{n\pi}{L}$, n=1,2,3,…….$\rightarrow \frac{\omega_n}{c} = \frac{n\pi}{L}$

Therefore, $y_n = B_n\sin\frac{n\pi}{L}x(C\cos\omega_n t + D\sin\omega_n t)$

General solution: $y = \sum_1^\infty y_n = \sum_1^\infty B_n\sin\frac{n\pi}{L}x(C_n\cos\omega_n t + D_n\sin\omega_n t)$

or, $y = \sum_1^\infty \sin\frac{n\pi}{L}x(a_n\cos\omega_n t + b_n\sin\omega_n t) = \sum_1^\infty \sin\frac{n\pi}{L}x\, c_n\cos(\omega_n t - \emptyset_n)$

Where, ${c_n}^2 = {a_n}^2 + {b_n}^2$ and $\tan\emptyset_n = \frac{a_n}{b_n}$

Thus, $\boldsymbol{y = \sum_1^\infty c_n \sin\frac{n\pi}{L}x\, \cos(\omega_n t - \emptyset_n)}$

Note:

$y_1 = c_1\sin\frac{\pi}{L}x\, \cos(\omega_1 t - \emptyset_1)$ gives displacement in Fundamental mode (n=1), 1st harmonic

$y_2 = c_2\sin\frac{2\pi}{L}x\, \cos(\omega_2 t - \emptyset_2)$ gives displacement in 2nd harmonic mode (n=2)

………

$y_n = c_n\sin\frac{n\pi}{L}x\, \cos(\omega_n t - \emptyset_n)$ gives displacement in nth harmonic mode

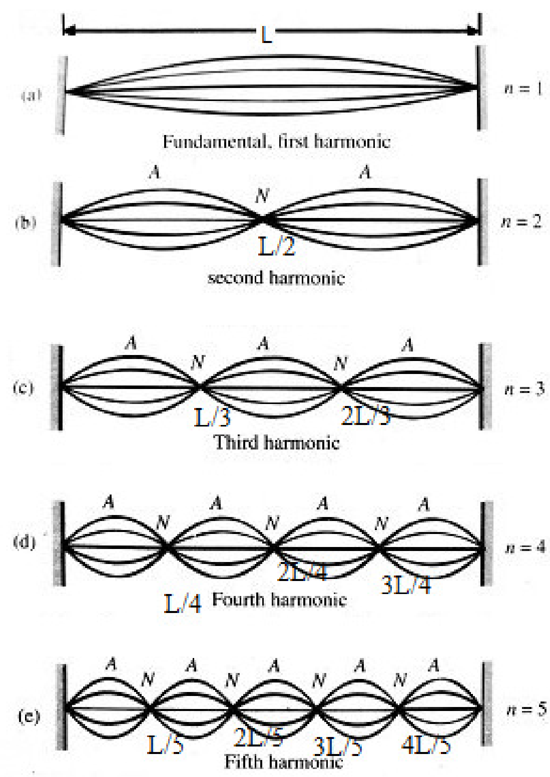
$$\frac{\omega_n}{c} = \frac{n\pi}{L} \to \omega_n = \frac{n\pi c}{L} \to \omega_n = \frac{n\pi}{L}\sqrt{\frac{F}{m}}$$

$$2\pi f_n = \omega_n = \frac{n\pi}{L}\sqrt{\frac{F}{m}} \to \boldsymbol{f_n} = \frac{\boldsymbol{n}}{\boldsymbol{2L}}\sqrt{\frac{\boldsymbol{F}}{\boldsymbol{m}}}$$

Therefore, frequency of fundamental mode, $\boldsymbol{f_1} = \frac{\boldsymbol{1}}{\boldsymbol{2L}}\sqrt{\frac{\boldsymbol{F}}{\boldsymbol{m}}}$

frequency of 1st harmonic mode, $\boldsymbol{f_2} = \frac{\boldsymbol{2}}{\boldsymbol{2L}}\sqrt{\frac{\boldsymbol{F}}{\boldsymbol{m}}} = \boldsymbol{2f_1}$

frequency of nth harmonic mode, $\boldsymbol{f_n} = \frac{\boldsymbol{n}}{\boldsymbol{2L}}\sqrt{\frac{\boldsymbol{F}}{\boldsymbol{m}}} = \boldsymbol{nf_1}$



It means, in the nth mode, dx element of string vibrates simple harmonically with amplitude $c_n\, sin\, \frac{n\pi}{L} x$ and angular frequency ω_n.

Try for string having both ends free

Energy of vibrating string

$$y = \sum_{1}^{\infty} c_n\, sin\, \frac{n\pi}{L} x\ \cos(\,\omega_n t - \emptyset_n\,)$$

For convenience, put $Y_n = c_n \cos(\,\omega_n t - \emptyset_n)$

Therefore, $\dot{Y_n} = c_n \omega_n \sin(\,\omega_n t - \emptyset_n)$

And $\dot{y} = \sum_1^\infty \dot{Y_n}\, sin\, \frac{n\pi}{L} x$

Total KE $= \frac{1}{2}\int_0^L m\, \dot{y}^2\, dx = \frac{m}{2}\int_0^L \left(\sum \dot{Y_n}\, sin\, \frac{n\pi}{L} x\right)^2 dx$

$$= \frac{m}{2}\int_0^L \sum_n \dot{Y_n}\, sin\, \frac{n\pi}{L} x \sum_m \dot{Y_m}\, sin\, \frac{m\pi}{L} x\ dx$$

$$= \frac{m}{2}\left(\int_0^L \sum_{n=m} \left(\dot{Y_n}\, sin\, \frac{n\pi}{L} x\right)^2 + 2\int_0^L \sum_{n\neq m} \dot{Y_n}\, sin\, \frac{n\pi}{L} x\, \dot{Y_m}\, sin\, \frac{m\pi}{L} x\right) dx$$

$$= \frac{m}{2}\frac{L}{2}\sum \dot{Y_n}^2 = \frac{M}{4}\sum \dot{Y_n}^2$$

M= total mass of the string

PE = WD in stretching dx to $ds = dx\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}$, tension remains const

$$= \int_0^L F\left[\left\{1 + \left(\frac{\partial y}{\partial x}\right)^2\right\}^{1/2} dx - dx\right] = \frac{F}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx$$

Now $y = \sum_1^\infty c_n\, sin\, \frac{n\pi}{L} x\ \cos(\,\omega_n t - \emptyset_n\,) = \sum Y_n\, sin\, \frac{n\pi}{L} x$

Therefore, $\frac{\partial y}{\partial x} = \sum Y_n \frac{n\pi}{L} cos \frac{n\pi}{L} x$

So, PE$= \frac{F}{2}\int_0^L \left(\sum Y_n \frac{n\pi}{L} cos \frac{n\pi}{L} x\right)^2 dx = \frac{F}{2}\int_0^L \sum {Y_n}^2 \left(\frac{n\pi}{L}\right)^2 cos^2 \frac{n\pi}{L} x\, dx$

$$= \frac{F}{2}\int_0^L \sum Y_n{}^2 \left(\frac{n\pi}{L}\right)^2 cos^2\frac{n\pi}{L}x\,dx = \frac{F}{2}\sum Y_n{}^2\left(\frac{n\pi}{L}\right)^2\frac{L}{2}$$

$$= \left(\frac{2Lf_n}{n}\right)^2\frac{m}{2}\sum Y_n{}^2\left(\frac{n\pi}{L}\right)^2\frac{L}{2} \quad \text{since} f_n = \frac{n}{2L}\sqrt{\frac{F}{m}}$$

$$= \sum mlY_n{}^2(\pi f_n)^2 = \sum MY_n{}^2\left(\frac{2\pi f_n}{2}\right)^2$$

$$= \sum MY_n{}^2\frac{\omega_n{}^2}{4} = \frac{M}{4}\sum Y_n{}^2\omega_n{}^2$$

$$\text{KE + PE} = \frac{M}{4}\sum \dot{Y_n}^2 + \frac{M}{4}\sum Y_n{}^2\omega_n{}^2 = \frac{M}{4}\left(\sum \dot{Y_n}^2 + \sum Y_n{}^2\omega_n{}^2\right)$$

$$= \frac{M}{4}\left(\sum(c_n\omega_n \sin(\omega_n t - \emptyset_n))^2 + \sum(c_n\omega_n \cos(\omega_n t - \emptyset_n))^2\right)$$

Therefore, total energy= $\boldsymbol{\frac{M}{4}\sum c_n{}^2\omega_n{}^2}$

Plucking and strucking strings

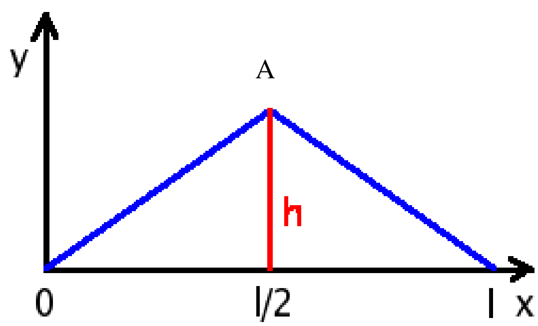
The strings of a musical instrument can either be plucked or struck in order to begin a vibration, producing -- a note. A guitar player plucks strings but a piano player makes little stricks(hit) strings inside the instrument. The difference is that the free oscillation begins when the guitar player releases the string in a position away from its rest (equilibrium) position, while the hammer in the piano hits the relaxed string. Similarly, the guitar string begins to accelerate towards the relaxed position, while the piano string is at maximum velocity as soon as the hammer hits it and then decelerates as it moves towards its amplitude.

Like a pendulum, the string oscillates between states of maximum kinetic energy (when it passes through the relaxed position) and maximum potential energy (amplitude). When a string is plucked, it begins its oscillation in the potential-energy state; if it is struck, it begins in the kinetic-energy state.

General solution of wave equation for stretched string fixed at both ends is $y = \sum_1^\infty sin\ \frac{n\pi}{L} x(a_n cos\ \omega_n t + b_n \sin \omega_n t)$

and $\dot{y} = \sum_1^\infty \omega_n\ sin\ \frac{n\pi}{L} x(-a_n sin\ \omega_n t + b_n \cos \omega_n t)$

For plucked string (guitar, say), at max. amplitude (at A): y(x,0)=f(x) and no motion before release: $\left.\frac{\partial y}{\partial x}\right|_{t=0} = 0$



Therefore, $\dot{y} = \sum_1^\infty \omega_n\ sin\ \frac{n\pi}{L} x(-a_n sin\ \omega_n.0 + b_n.0) = 0 \rightarrow \quad b_n = 0$

So $\quad y = \sum_1^\infty a_n sin\ \frac{n\pi}{L} x\ \ cos\ \omega_n t$

At t=0, $y_0 = \sum_1^\infty a_n sin\ \frac{n\pi}{L} x$

For struck string,

at relaxed position: y(x,0)=0

and max velocity at release $\left.\frac{\partial y}{\partial x}\right|_{t=0} = V(x)$

Therefore, $\quad y = \sum_1^\infty sin\ \frac{n\pi}{L} x(a_n cos\ \omega_n.0 + b_n.0) = 0 \rightarrow \quad a_n = 0$

So, $y = \sum_1^\infty b_n sin\,\frac{n\pi}{L}x\ \sin\omega_n t$

At t=0, $y_0 = \sum_1^\infty b_n sin\,\frac{n\pi}{L}x\ \sin\omega_n t$, and $\dot{y}_0 = \sum_1^\infty b_n\omega_n\, sin\,\frac{n\pi}{L}x$

Now what remains is to calculate a_n for plucked string and b_n **for struck string.**

This is done using following equations.

$\boldsymbol{a_n = \frac{2}{L}\int_0^L y_0 sin\frac{n\pi x}{L}dx}$ **and** $\boldsymbol{b_n = \frac{2}{L\omega_n}\int_0^L \dot{y}_0 sin\frac{n\pi x}{L}dx}$

Try ………..

Ref: Advanced Acoustics by DP Raychaudhuri