SUPERPOSITION OF SHMs

TOPICS:

Principle of Superposition

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PRINCIPLE OFSUPERPOSITION

"*The resultant of two or more harmonic displacements is simply the vector sum of the individual displacements*."

SUPERPOSITION OF TWO COLLINEAR HARMONIC OSCILLATIONS

1. Oscillations Having Equal Frequencies

Suppose we have two SHMs of equal frequencies but having different amplitudes and different phase constants acting on a system in the x-direction. The displacements x_1 and x_2 of the two harmonic motions, of the same angular frequency ω, differing by phase δ are given by

$$x_1 = A_1 sin\omega t$$

$$x_2 = A_2 sin(\omega t + \delta)$$

There are two methods which can be used to obtain an expression for the resultant displacement due to superposition of the above two harmonic oscillations. Let us discuss them now.

(a) *<u>Analytical Method</u>*

We use the superposition principle which states that the resultant displacement equal to the vector sum (algebraic sum in this case, because the direction of the two individual oscillation is in the x-direction) of the individual displacements. Therefore, we can write

$$x = x_1 + x_2$$

$$x = A_1 sin\omega t + A_2 sin(\omega t + \delta)$$

$$= A_1 sin\omega t + A_2 sin\omega t\ cos\delta + A_2 cos\omega t\ sin\delta \text{ ======}$$

$$= (A_1 + A_2 \cos\delta)\ sin\omega t + (A_2 sin\delta) cos\omega t$$

$$= C\ sin\omega t + D \cos\omega t$$

where $C = (A_1 + A_2 \cos\delta)$ $and\ D = (A_2 sin\delta)$

$$x = \sqrt{C^2 + D^2}\left[\frac{C}{\sqrt{C^2 + D^2}} sin\omega t + \frac{D}{\sqrt{C^2 + D^2}} cos\omega t\right]$$

Put, $sin\alpha = \frac{D}{\sqrt{C^2+D^2}}$ and $cos\alpha = \frac{C}{\sqrt{C^2+D^2}}$

Therefore, we get $x = \sqrt{C^2 + D^2}[cos\alpha\ sin\omega t + sin\alpha\ cos\omega t]$

$$x = Asin(\omega t + \alpha)$$

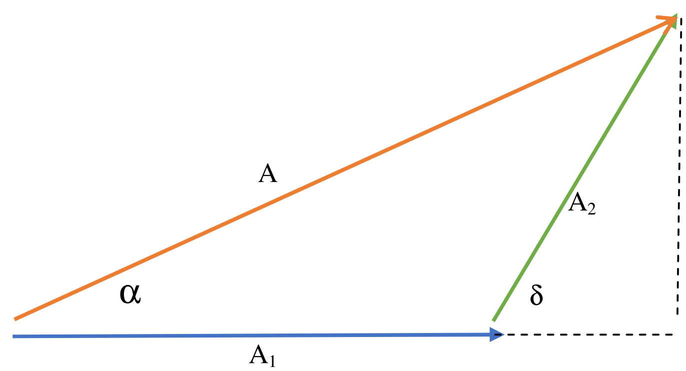
$$A = \sqrt{C^2 + D^2} = \sqrt{(A_1 + A_2 cos\delta)^2 + (A_2 sin\delta)^2} = \sqrt{A_1^2 + A_2^2 + A_1 A_2 cos\delta}$$

$$tan\alpha = \frac{D}{C} = \frac{A_2 sin\delta}{A_1 + A_2 cos\delta}$$

The resultant of two collinear simple harmonic motions having the same frequency is itself a simple harmonic motion. The amplitude and phase of the resultant SHM depends on the amplitudes of the two individual SHMs as well as the phase difference between them.

(b) *VectorMethod*

SHM is represented as rotating vector. Represent the first SHM by a vector of magnitude $A_1$and the second SHM by the vector of magnitude A_2. We assume here that the phase angle of the first vector is zero and for the A second vector it isδ. Therefore the phase difference between them is δ.



The resultant oscillation : $x = Asin(\omega t + \alpha)$

The vector A˙ represents the resultant SHM whose magnitude is

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2cos\delta}$$

The resultant A˙ makes an angle αwith the x-axis and can be expressed as

$$tan\alpha = \frac{Height}{Base} = \frac{A_2 sin\delta}{A_1 + A_2 cos\delta}$$

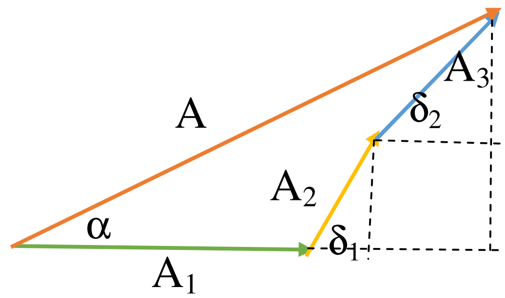
The advantage with the vector method is that it can easily be extended to more than two vectors. For example, if we have 3 vectors,

$$x_1 = A_1 sin\omega t$$

$$x_2 = A_2 sin(\omega t + \delta_1)$$

$$x_3 = A_2 sin(\omega t + \delta_2)$$

they can be represented as shown in the figure below.



The resultant oscillation : $x = Asin(\omega t + \alpha)$

The vector A˙ represents the resultant SHM whose magnitude is

$$A = \sqrt{(A_1 + A_2 cos\delta_1 + A_3 cos\delta_2)^2 + (A_2 sin\delta_1 + A_2 sin\delta_2)^2}$$

The resultant A˙ makes an angle αwith the x-axis and can be expressed as

$$tan\alpha = \frac{Height}{Base} = \frac{A_2 sin\delta_1 + A_2 sin\delta_2}{A_1 + A_2 cos\delta_1 + A_3 cos\delta_2}$$

APPLICABLE FOR N VECTORS ALSO

Q: A particle is subjected to two simple harmonic oscillations

$$x_1 = Asin\,\omega t$$
$$x_2 = Asin\left(\omega t + \frac{\pi}{3}\right)$$

Determine (a) the displacement at t = 0, (b) the maximum speed of the particle and (c) the maximum acceleration of the particle

Q. Calculate the amplitude and initial phase of the harmonic oscillations obtained by the superposition of two collinear oscillations represented by the following equations:

$$x_1 = 0.02N sin\left(5\pi t + \frac{\pi}{2}\right)$$
$$x_2 = 0.03N sin\left(5\pi t + \frac{\pi}{4}\right)$$

Oscillations Having Different Frequencies:Beats

Suppose we have two collinear harmonic oscillations of different frequencies and different amplitudes and the same phase constant (= zero).

$$x_1 = A_1 sin\ \omega_1 t$$
$$x_2 = A_2 sin\ \omega_2 t$$

The resultant of these two oscillation is given by

$$x = A_1 sin\ \omega_1 t + A_2 sin\ \omega_2 t$$

Define $\omega_1 = \omega - \Delta\omega$ and $\omega_1 = \omega + \Delta\omega$

Therefore $\omega = \frac{1}{2}(\omega_1 + \omega_2)$ Av. frequency and
$\Delta\omega = \frac{1}{2}(\omega_1 - \omega_2)$ modulation frequency

Substituting,

$$x = A_1 \sin\omega_1 t + A_2 \sin\omega_2 t$$

$$= A_1 sin\ (\omega - \Delta\omega)t + A_2 sin\ (\omega + \Delta\omega)t$$

$$= A_1(\sin\omega t\ cos\Delta\omega t - cos\omega t\ sin\Delta\omega t) + A_2(\sin\omega t\ cos\Delta\omega t + cos\omega t\ sin\Delta\omega t)$$

$$= (A_1 + A_2)\sin\omega t\ cos\Delta\omega t - (A_1 - A_2) cos\omega t\ sin\Delta\omega t$$

Define

$$(A_1 + A_2) cos\Delta\omega t = A cos\alpha$$

$$-(A_1 - A_2) sin\Delta\omega t = A\ sin\alpha$$

$$x = Acos\alpha\ sin\omega t + Asin\alpha\ cos\omega t = Asin(\omega t + \alpha)$$

A= Resultant Amplitude, α= Resultant Phase const. This oscillation can, at best, be described as periodic with an angular frequency of $\omega = \frac{1}{2}(\omega_1 + \omega_2)$, the average of the two componentfrequencies.

$$A = \sqrt{(A_1 + A_2)^2 cos^2\Delta\omega t + (A_1 - A_2)^2 sin^2\Delta\omega t}$$

$$= \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2 \cos(2\Delta\omega t)}$$

And $\tan\alpha = \frac{-(A_1-A_2)\sin\Delta\omega t}{(A_1+A_2)\cos\Delta\omega t}$

Both A and α vary with time

If $\omega_1 \approx \omega_2$, $\Delta\omega \to 0$, modulated frequency $\Delta\omega \ll \omega$, average frequency. Modulated amplitude (resultant amplitude) A and the modulated phase constant (resultant phase constant) vary only slightly with time and may be treated as almost constant. Then

$$x = A\sin(\omega t + \alpha)$$

represent an approximate harmonic oscillations having angular frequency ω. The resulting oscillations, in the case when the two frequencies of the SHMs are nearly equal, exhibit what are called beats.

Q When is the modulated amplitude $A = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2\cos(2\Delta\omega t)}$ maximum?

Solution: maximum modulated amplitude $=A_1 + A_2$
when $\cos(2\Delta\omega t) = 1 \to 2\Delta\omega t = 0, 2\pi$ ....

$$\to (\omega_2 - \omega_1)t = 0, 2\pi \ldots 2n\pi$$

$$\to t = 0, \frac{2\pi}{(\omega_2 - \omega_1)}, \frac{4\pi}{(\omega_2 - \omega_1)} \ldots \ldots$$

$$\to t = 0, \frac{1}{(\nu_2 - \nu_1)}, \frac{2}{(\nu_2 - \nu_1)} \ldots \ldots$$

Hence the time interval between two consecutive e maxima is is $\frac{1}{\nu_2 - \nu_1}$

Q: When is the modulated amplitude minimum?

the minimum amplitude = $|A_1 - A_2|$, when
when $\cos(2\Delta\omega t) = -1 \to 2\Delta\omega t = \pi, 2\pi$ ....

$$\to (\omega_2 - \omega_1)t = \pi, 3\pi, 5\pi \ldots$$

$$\to t = \frac{\pi}{(\omega_2 - \omega_1)}, \frac{3\pi}{(\omega_2 - \omega_1)}, \frac{5\pi}{(\omega_2 - \omega_1)} \ldots \ldots$$

$$\rightarrow t = \frac{1}{2(\nu_2 - \nu_1)}, \frac{3}{2(\nu_2 - \nu_1)}, \frac{5}{2(\nu_2 - \nu_1)} \dots \dots$$

Hence the time interval between two consecutive e minima is is $\frac{1}{\nu_2 - \nu_1}$

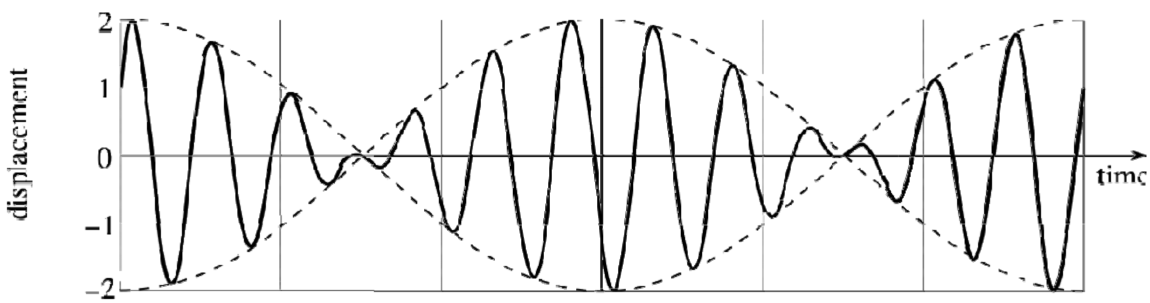
The periodic variation of the amplitude of the motion, resulting from the superposition of SHMs of slightly different frequencies, is known as the phenomenon of beats. A maxima followed by a minima is technically called a beat. The time period t_b between the successive beats is called the beat period given by

$$t_b = \frac{1}{\nu_2 - \nu_1}$$

And the beat frequency ν_b is given by $|\nu_2 - \nu_1|$

Hence, the beat frequency is equal to the difference between the frequencies of the component oscillations.

The figure below displays graphically the result of superimposing two harmonic oscillations of different frequencies. You may note that superposition of two SHMs of different frequencies results in oscillations that are periodic but not harmonic.



Application of Beats

The phenomenon of beats is of great importance. Beats can be used to determine the small difference between frequencies of two sources of sound. Musicians often make use of beats in tuning their instruments. If the instrument is out of tune, one will hear beats. Sometimes beats are deliberately produced in a particular section of an orchestra to give a pleasing tone tothe resulting sound. There are many physical phenomena which involve beats.

Q: Two radio stations broadcast their programmes at the same amplitude A, and at slightly different frequencies, where their difference is equal to 10^3 Hz. A detector receives the signals from the two stations simultaneously. Find the time interval between successive maxima of the intensity of the signal received by the detector.

The time interval between successive maxima of the intensity is equal to the beat period

$$t_b = \frac{1}{\nu_2 - \nu_2} = 10^{-3}$$

When two tuning forks of same frequency are sounded, a continuous sound is heard. When one of the tuning forks is waxed a little, so as to reduce its frequency, beats are heard because now the frequencies of the two tuning forks are slightly different. By counting the number of beats heard in a given interval of time, one can calculate the beat frequency.

Q: A tuning fork A produces 4 beats with tuning fork B of frequency 256 Hz. When A is waxed, the beats are found to occur at shorter intervals. What was its original frequency?

Solution: As the tuning fork A produces 4 beats with B of frequency 256 Hz, from equation (4.15), the frequency of A should be

$$\upsilon_1=(\nu_2\pm\nu_b)=256\pm4\text{Hz}$$

Now when tuning fork A is waxed, we are given that the beat period $t_b = \frac{1}{\nu_2 - \nu_2}$ *decreases*

Or, in other words, $\nu_b = |\nu_2 - \nu_1|$ increases.

Let the frequency of the tuning fork A is υ_1=256+4=260Hz.WhentuningforkAis waxed,itsfrequency$\upsilon_1$willdecrease,andhencethebeatfrequencyshouldalsodecrease.Butit isnotwhatwehavefoundabove.So,frequencyofAcannotbe260Hz.

Let υ_1=256–4=252Hz, then its frequency $\upsilon_1$decreases due to waxing, the beat frequency increases.Hence,υ_1=252Hzisthecorrectanswer.

Example 1: Show that if a particle is acted upon by two separate forces each of which can separately produce a simple harmonic motion, the resultant motion of the particle is a combination of two simple harmonicmotions

Example 2: What is the maximum possible amplitude of the two simple harmonic motions and when does it occur?

Example 3:What is the minimum possible amplitude of two simple harmonic motions and when does it occur?

Ref: Advanced Acoustics by D P Raychowdhuri

Vibration waves and acoustics by D Chattopadhyay and P C Rakshit