Syllabus : Superposition of two perpendicular oscillations for phase difference 0, π/2, π, Graphical and analytical methods, Lissajous figures with equal and unequal frequencies

TWO MUTUALLY PERPENDICULAR HARMONIC OCILLATIONS HAVING SAME FREQUENCIES

Let's assume that two independent forces acting on a particle in such a manner that the first alone produces a simple harmonic motion in the x- direction givenby

$x = A_1 sin\omega t$

and the second would produce a simple harmonic motion in the y-direction given by

$y = A_2 sin(\omega t + \delta)$

Thus, we are actually considering the superposition of two mutually perpendicular SHMs which have equal frequencies. The amplitudes are different and their phases differ byδ. The resultant motion of the particle is a combination of the twoSHMs.

The position of the particle at any time t is given by (x, y) where x and y are given by the above equations. The *resultant motion is, thus, two-dimensional* and the path of the particle is, in general, an ellipse. The equation of the path traced by the particle is obtained by eliminating t from above equations

Now,

$sin\omega t = \frac{x}{A_1}$ and $cos\omega t = \sqrt{1 - \frac{x^2}{{A_1}^2}}$

Putting these values in equation for y,

$$y = A_2(sin\omega t\ cos\delta + cos\omega t\ sin\delta) = A_2\left(\frac{x}{A_1}cos\delta + \sqrt{1 - \frac{x^2}{{A_1}^2}}\ sin\delta\right)$$

Or, $$\frac{y}{A_2} = \left(\frac{x}{A_1}cos\delta + \sqrt{1 - \frac{x^2}{{A_1}^2}}\ sin\delta\right)$$

Or, $$\frac{y}{A_2} - \frac{x}{A_1}cos\delta = \sqrt{1 - \frac{x^2}{{A_1}^2}}\ sin\delta$$

Or, $$\left(\frac{y}{A_2} - \frac{x}{A_1}cos\delta\right)^2 = \left(\sqrt{1 - \frac{x^2}{{A_1}^2}}\ sin\delta\right)^2$$

Or, $$\frac{y^2}{{A_2}^2}+\frac{x^2}{{A_1}^2}\cos^2\delta-\frac{2xy}{A_1A_2}\cos\delta=\left(1-\frac{x^2}{{A_1}^2}\right)\sin^2\delta$$

Or, $$\frac{y^2}{{A_2}^2}+\frac{x^2}{{A_1}^2}\cos^2\delta+\frac{x^2}{{A_1}^2}\sin^2\delta-\frac{2xy}{A_1A_2}\cos\delta=\sin^2\delta$$

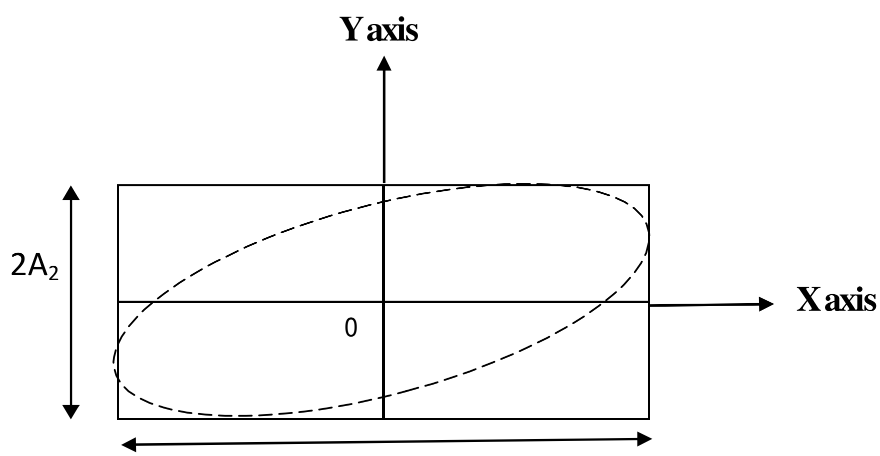
Or, $$\frac{y^2}{{A_2}^2}+\frac{x^2}{{A_1}^2}-\frac{2xy}{A_1A_2}\cos\delta=\sin^2\delta \qquad \text{------------------------(1)}$$

This is an equation of ellipse. Thus, we may conclude that the resultant motion of the particle is along an elliptical path.

Here x remains between $-A_1$ and A_1 and that of y remains between $-A_2$ and A_2. Thus, the particle always remains inside the rectangle defined by

$x = \pm A_1$ and $y = \pm A_2$

The ellipse can be drawn as,



Special Cases

The two component SHMs are in phase, $\delta = 0$

The two component SHMs are out of phase, $\delta = \pi$

Thephasedifferencebetweenthetwocomponent SHMs, $\delta = \pi/2$

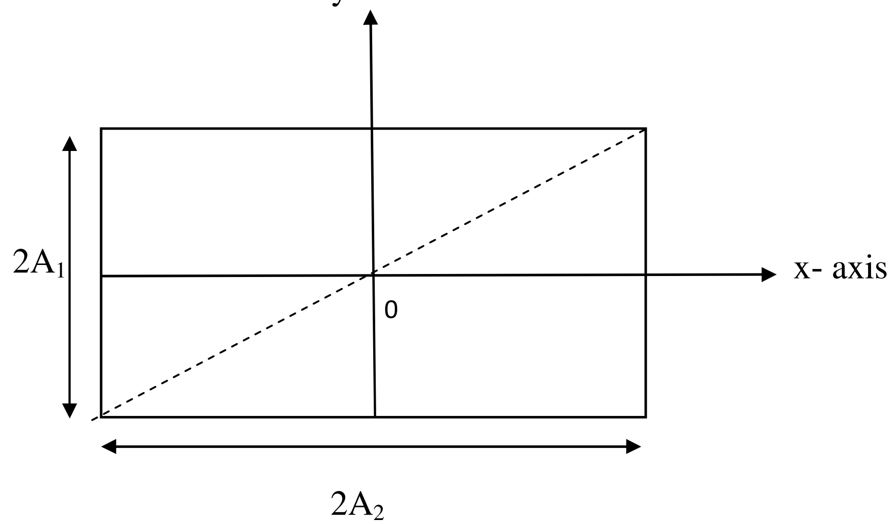
Let us now obtain the resultant motion of the particle under these special cases.

When the two superposing SHMs are in phase, $\delta = 0 \Rightarrow \cos\delta = 1, \quad \sin\delta = 0$

Eq(1) : $$\frac{y^2}{{A_2}^2}+\frac{x^2}{{A_1}^2}-\frac{2xy}{A_1A_2}=0 \quad \Rightarrow\left(\frac{y}{A_2}-\frac{x}{A_1}\right)^2=0 \quad \Rightarrow\frac{y}{A_2}-\frac{x}{A_1}=0$$

This is an equation of a straight line passing through the origin and have a slope of$tan^{-1}\left(\frac{A_2}{A_1}\right)$

The path of the particle is:



For δ=0, $x = A_1 sin\omega t,\quad y = A_2 sin\omega t \Rightarrow r = \sqrt{{A_1}^2 + {A_2}^2}\, sin\omega$

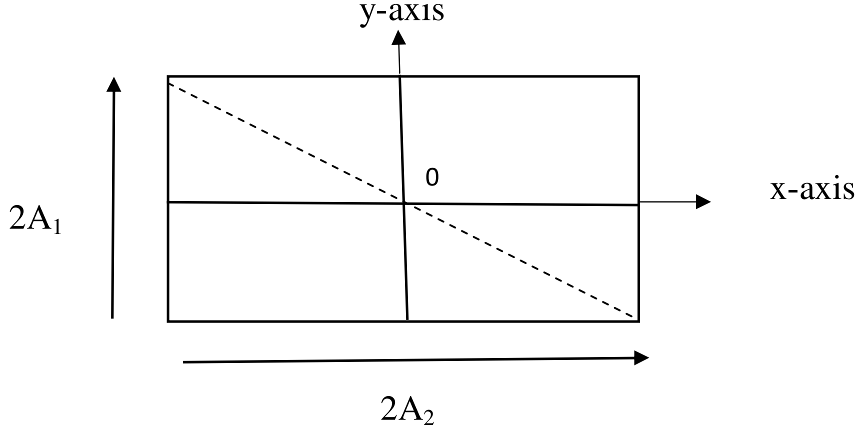
Thus, we can see that the resultant motion is also SHM with the same frequency and phase as the component motions. However, the amplitude of the resultant SHM is $\sqrt{{A_1}^2 + {A_2}^2}$

When the two superposing SHMs are in phase, δ =$\pi \Rightarrow\ cos\delta = -1,\ sin\delta = 0$

Eq(1) : $\frac{y^2}{{A_2}^2} + \frac{x^2}{{A_1}^2} + \frac{2xy}{A_1 A_2} = 0 \quad \Rightarrow \left(\frac{y}{A_2} + \frac{x}{A_1}\right)^2 = 0 \quad \Rightarrow \frac{y}{A_2} + \frac{x}{A_1} = 0$

This is an equation of a straight line passing through the origin and have a slope of $tan^{-1}\left(-\frac{A_2}{A_1}\right)$

The path of the particle is:

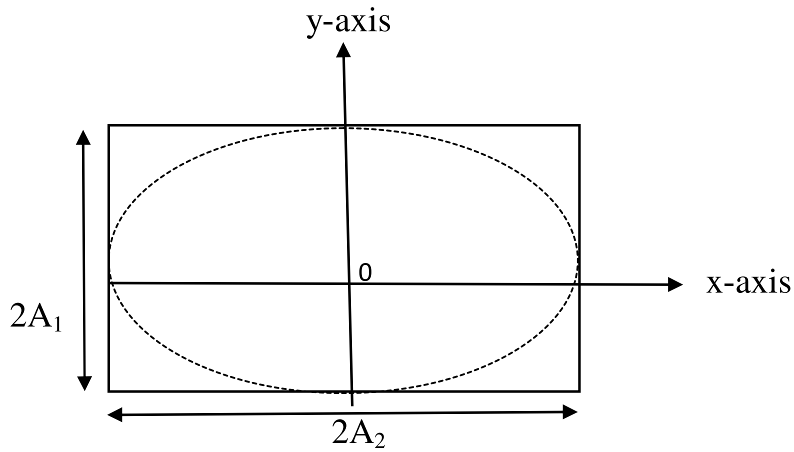


WhenthephasedifferencebetweenthetwocomponentSHMsis$\delta = \pi/2$.

$$\Rightarrow\ cos\delta = 0,\quad sin\delta = 1$$

Eq(1) : $$\frac{y^2}{{A_2}^2} + \frac{x^2}{{A_1}^2} = 1$$

This is an equation of ellipse with its axes along the x-axis and the y- axis and with its center at the origin. The lengths of the major and the minor axes are $2A_1$and $2A_2$,respectively.. The particle will move in an ellipse.



In case the amplitudes of the two individual SHMs are equal, $A_1 = A_2 = A$, i.e. the major and the minor axes are equal, then the ellipse reduces to acircle.

$$x^2 + y^2 = A^2$$

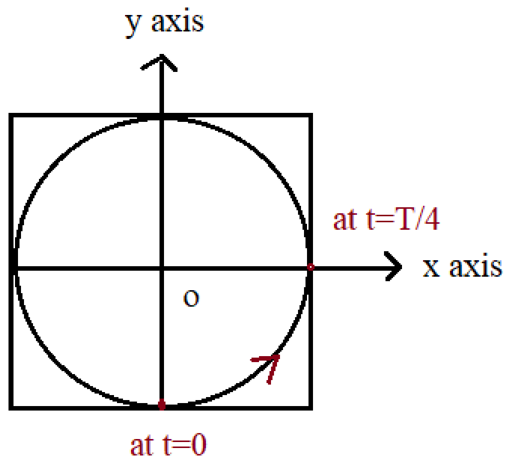
Thus, the resultant motion of a particle due to superposition of two mutually perpendicular SHMs of equal amplitude and having a phase difference of π/2 is a *circular motion*. The circular motion may be clockwise or anticlockwise depending on which component leads theother.

Example 1: Show that the superposition of oscillations represented by $x = A sin\omega t$, $y = -A cos\omega t$ results in to circular motion traced in the anticlockwise sense.

Solution: Here phase difference $\delta = 0$, particle moves in path $x^2 + y^2 = A^2$, which is a circle of radius A.

At t=0, $x = 0$, $y = -A$

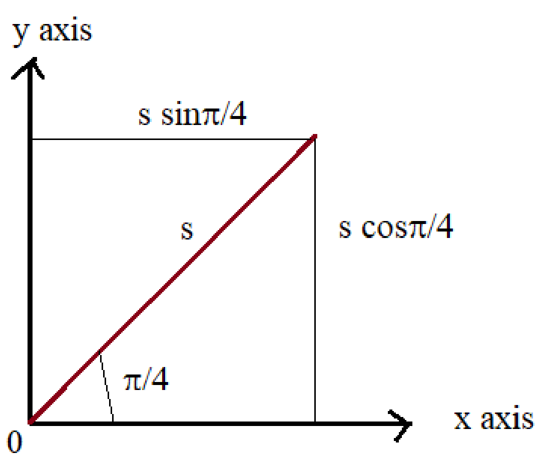
At t=T/4, $x = \frac{Asinw2\pi}{4w} = \frac{Asin\pi}{2} = A$, $y = -Acosw2\pi/4w = -Acos\pi/2 = 0$



i.e motion of the particle is anticlockwise.

Example 2: A particle is subjected to two simple harmonic oscillations, one along the x-axis and the other on a line making an angle of π/4 with the x-axis. The two motions are given by x=Asinωt and S=Bsinωt. Calculate the amplitude of the resultant motion.

Solution: The two individual SHMs, with equal frequencies, phase difference between themδ=0. Now, let's break the second SHM into its x-component and the y-component.



Components along X- axis : $\frac{S}{\sqrt{2}} + A\ sin\omega t = \frac{B\sin\omega t}{\sqrt{2}} + A\ sin\omega t = \left(A + \frac{B}{\sqrt{2}}\right) sin\omega t$

Component along Y-axis: $\frac{S}{\sqrt{2}} = \frac{B\sin\omega t}{\sqrt{2}}$

The amplitude of the resultant oscillation for mutually perpendicular oscillations with zero phase difference is given by

$$= \sqrt{\left(A + \frac{B}{\sqrt{2}}\right)^2 + \left(\frac{B}{\sqrt{2}}\right)^2} = \sqrt{A^2 + \frac{B^2}{2} \times 2 + \frac{2}{\sqrt{2}}AB} = \sqrt{A^2 + B^2 + \sqrt{2}AB}$$

Exercise:

1. Show that the superposition of oscillations represented by =Asinωt and y=Acosωt results in to circular motion traced in the clockwise sense.
2. A particle is subjected to two simple harmonic motions in the perpendicular directions having equal amplitudes (0.01 m) and equal frequencies. If the two SHMs are out of phase, what is the nature of the path followed by the particle?
3. A particle is subjected to two simple harmonic motions in the perpendicular direction having equal amplitudes (0.01 m) and equal frequencies. If the two component SHMs are in phase, what is the nature of path followed by the particle?
4. A particle is subjected to two SHMs y=sinωt and z=sinωt. What is the path that the particle follows?
5. A particle is subjected to two simple harmonic oscillations, one along the x-axis and the other on a line making an angle of 3π/4 with the x-axis. The two motions are given by x =Asinωt and S=Bsinωt. Calculate the amplitude of the resultant motion.

Oscillations Having Different Frequencies (Lissajous Figures)

When the frequencies of the two perpendicular SHMs are not equal, the resulting motion becomes more complicated. Let us suppose that the displacements of the two mutually perpendicular oscillations are givenby

$$x = A_1 sin\omega_1 t$$

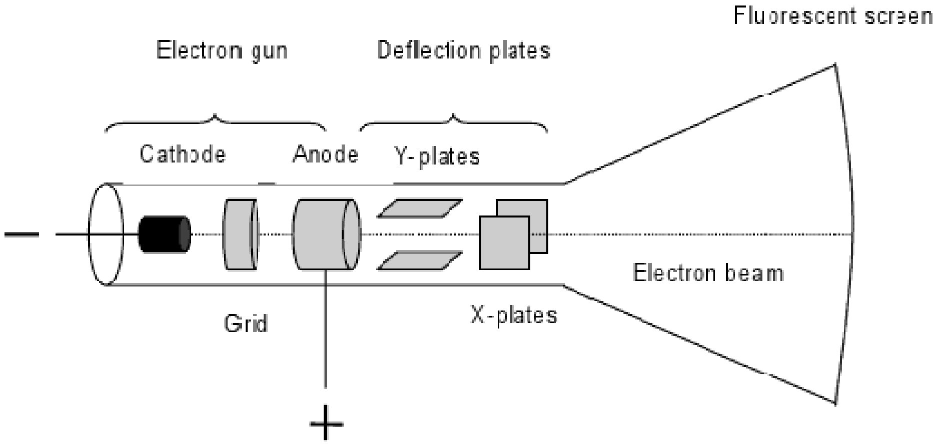
$$y = A_2 sin(\omega_2 t + \delta)$$

The phase difference between them at any instant t, is given by

$$\Delta = (\omega_2 t + \delta) - \omega_1 t = (\omega_2 - \omega_1)t + \delta$$

Since the superimposed orthogonal oscillations are of different frequencies, one of them changes faster than the other and will gain in phase over the other. As a result, the resultant motionpassesthroughdifferentphases.The phase of resultantchangeswithtimeduetothechangeinthephasedifference of superimposed oscillations.However, the motion is confined within a rectangle of sides $2A_1$ and $2A_2$.

The pattern of the resultant motion is described by a *Lissajous figure*, named after French mathematician. **Lissajous figures can be seen by using a cathode ray oscilloscope (CRO)**.



Here, two rectangular oscillations are simultaneously imposed upon a beam of cathode ray byconnecting two sources of electrical oscillations to horizontal plates XX and vertical plates YY of the oscilloscope. We then see the trace of the resultant effect in the form of an electron beam on the fluorescent screen. By adjusting the phases, amplitudes and the ratio of the frequencies of the applied voltage, we obtain various curves. Lissajous figures may be used to compare two nearly equal frequencies. If the frequencies of two component oscillations are not exactly equal, the Lissajous figure will change gradually.

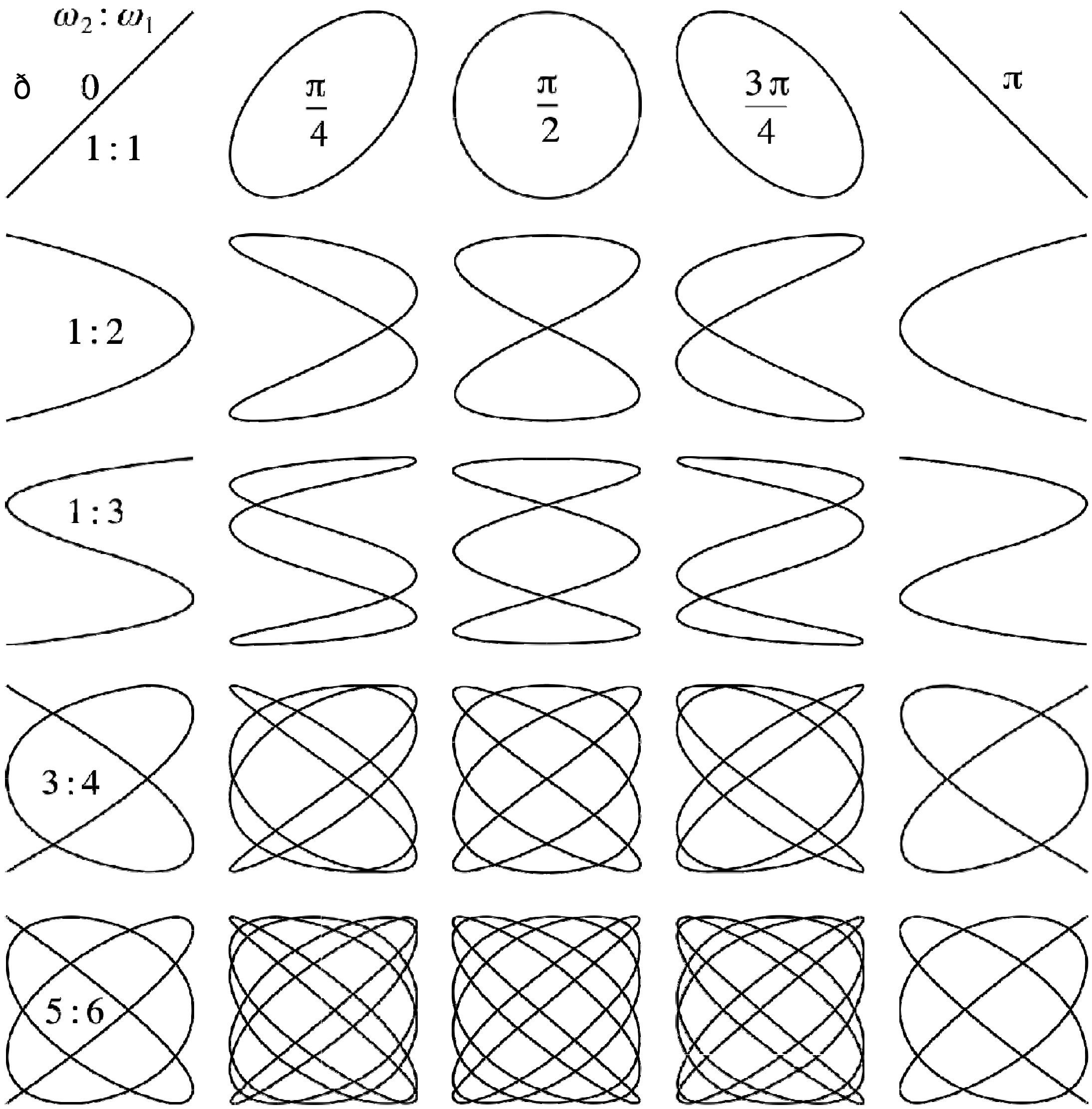
Let us now look at few examples to illustrate the shape of the Lissajous figure for some special cases.

Suppose a particle is subjected to two mutually perpendicular simple harmonic oscillations,

$$x = A_1 cos\omega_1 t$$
$$y = A_2 \cos(\omega_2 t + \delta)$$

If you try to trace the trajectory of the motion of the particle using analytical methodi.e to find y in terms of x, eliminating t, calculation becomes very cumbersome if the phase difference δ is nonzero. Below is a table showing lissajous figure for δ=0, ω_1 and ω_2 varying in the ratio 1:1(both equal), 1:2 ($\omega_2=2\omega_1$), 1:3($\omega_2=3\omega_1$)………………

Here amplitudes are taken equal ($A_1 = A_2$)



Example 3: A particle is subjected to two mutually perpendicular simple harmonic oscillations,

$$x = A_1 cos\omega t$$
$$y = A_2 \cos{(2\omega t + \delta)}$$

Here ratio of frequency is 1:2

Eliminating t from these two equations, we can determine the equation of the trajectory of the particle , i.e. we are looking for an expression for y in terms ofx.

$$cos\omega t = \frac{x}{A_1}$$

Now, $y = A_2 \cos(2\omega t + \delta) = A_2[\cos 2\omega t\ cos\delta - \sin 2\omega t\ sin\delta]$

$$= A_2[(2cos^2\omega t - 1)\ cos\delta - 2\sin\omega t\ cos\omega t\ sin\delta]$$

$$= A_2\left[\left(2\frac{x^2}{{A_1}^2} - 1\right)cos\delta - 2\frac{x}{A_1}\sqrt{1 - \frac{x^2}{{A_1}^2}}\ \ sin\delta\right]$$

$$\frac{y}{A_2} = \left(2\frac{x^2}{{A_1}^2} - 1\right)cos\delta - 2\frac{x}{A_1}\sqrt{1 - \frac{x^2}{{A_1}^2}}\ \ sin\delta$$

$$\frac{y}{A_2} + cos\delta - \left(2\frac{x^2}{{A_1}^2}\right)cos\delta = -2\frac{x}{A_1}\sqrt{1 - \frac{x^2}{{A_1}^2}}\ \ sin\delta$$

Squaring,

$$\left(\frac{y}{A_2} + cos\delta\right)^2 + \left(2\frac{x^2}{{A_1}^2}\right)^2 cos^2\delta - 2\left(\frac{y}{A_2} + cos\delta\right)2\frac{x^2}{{A_1}^2}cos\delta = \left(2\frac{x}{A_1}\sqrt{1 - \frac{x^2}{{A_1}^2}}\ \ sin\delta\right)^2$$

Or, $\left(\frac{y}{A_2} + cos\delta\right)^2 + 4\left(\frac{x^2}{{A_1}^2}\right)^2 cos^2\delta - 4\left(\frac{y}{A_2} + cos\delta\right)\frac{x^2}{{A_1}^2}cos\delta = 4\left(\frac{x}{A_1}\right)^2\left(1 - \frac{x^2}{{A_1}^2}\right)sin^2\delta$

$$or, \left(\frac{y}{A_2} + cos\delta\right)^2 + 4\frac{x^4}{{A_1}^4}cos^2\delta - 4\left(\frac{y}{A_2} + cos\delta\right)\frac{x^2}{{A_1}^2}cos\delta$$
$$= 4\left(\frac{x}{A_1}\right)^2 sin^2\delta - 4\frac{x^4}{{A_1}^4}sin^2\delta$$

Or, $\left(\frac{y}{A_2} + cos\delta\right)^2 + 4\left(\frac{x^4}{{A_1}^4}\right)(cos^2\delta + sin^2\delta) - 4\left(\frac{y}{A_2} + cos\delta\right)\frac{x^2}{{A_1}^2}cos\delta = 4\left(\frac{x}{A_1}\right)^2 sin^2\delta$

Or, $\left(\frac{y}{A_2} + cos\delta\right)^2 + 4\left(\frac{x^4}{{A_1}^4}\right) - 4\left(\frac{y}{A_2} + cos\delta\right)\frac{x^2}{{A_1}^2}cos\delta = 4\left(\frac{x}{A_1}\right)^2 sin^2\delta$

Or, $\left(\frac{y}{A_2} + cos\delta\right)^2 + 4\left(\frac{x^2}{{A_1}^2}\right)\left[\frac{x^2}{{A_1}^2} - \left(\frac{y}{A_2} + cos\delta\right)cos\delta - sin^2\delta\right] = 0$

Or, $\left(\frac{y}{A_2}+cos\delta\right)^2+4\left(\frac{x^2}{A_1{}^2}\right)\left[\frac{x^2}{A_1{}^2}-\frac{y}{A_2}cos\delta-(cos^2\delta+sin^2\delta)\right]=0$

Or, $\boldsymbol{\left(\frac{y}{A_2}+cos\delta\right)^2+4\left(\frac{x^2}{A_1{}^2}\right)\left[\frac{x^2}{A_1{}^2}-\frac{y}{A_2}cos\delta-1\right]=0}$

The above equation is of fourth degree, which, in general, represents a closed curve having two loops. For a given value of δ, the curve corresponding to the above expression can be traced using the knowledge of coordinategeometry.

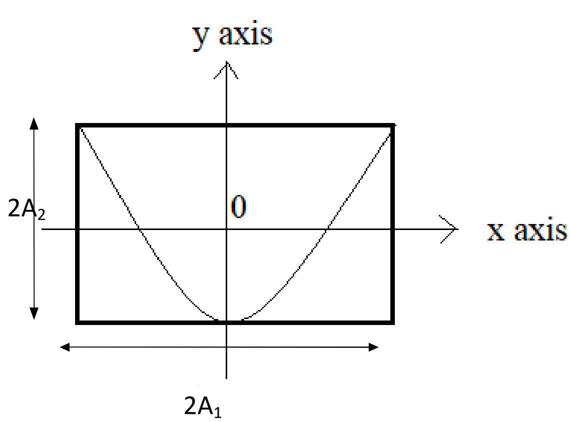
For $\delta=0, cos\delta=1$, we get $\left(\frac{y}{A_2}+1\right)^2+4\left(\frac{x^2}{A_1{}^2}\right)\left[\frac{x^2}{A_1{}^2}-\frac{y}{A_2}-1\right]=0$

Or, $$\left(\frac{y}{A_2}+1\right)^2+\left(\frac{2x^2}{A_1{}^2}\right)^2-2\left(\frac{2x^2}{A_1{}^2}\right)\left(\frac{y}{A_2}+1\right)=0$$

Or, $$\left(\frac{y}{A_2}+1-\frac{2x^2}{A_1{}^2}\right)^2=0$$

Or, $$\frac{y}{A_2}+1=\frac{2x^2}{A_1{}^2}$$

This represents two coincident parabolas with their vertices at $(0,-A_2)$.



Example4 :A particle is subjected to two mutually perpendicular simple harmonic oscillations,

$$x=2\cos t$$
$$y=\cos(t+4)$$

Trace the trajectory of the particle using graphical method.

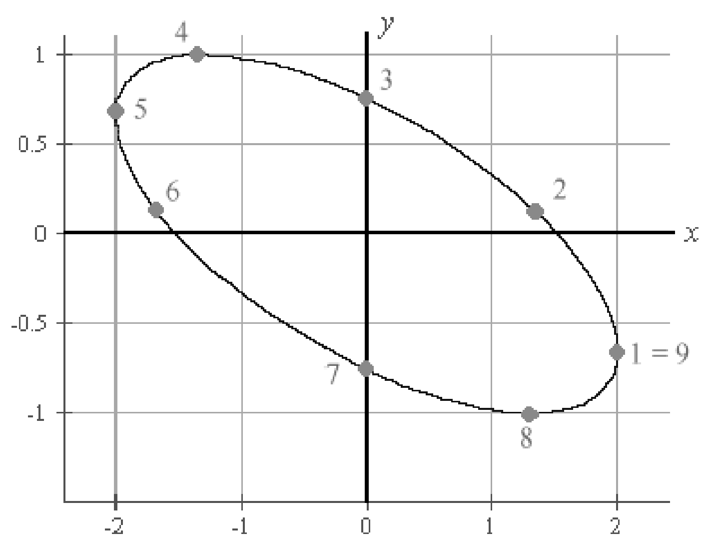
cases, the resultant The analytical solution becomes very cumbersome if the phase difference δ is non- zero. In such motion can be constructed quite conveniently by using graphical method.

Graphical method

Here, we set up a table of values to see what is happening. We give each point a "point number" so that it is easier to understand when we graph thecurve.

t	0	π/4	π/2	3π/4	π	5π/4	3π/2	7π/4	2π
x	2	1.4	0	-1.4	-2	-1.4	0	1.4	2
y	-0.6	0.1	0.7	1	0.7	-0.1	-0.8	-1	-0.7
Pt.no.	1	2	3	4	5	6	7	8	9

From the above table, the resulting curve, with the numbered points included, is shown in the following figure. Point 1 is actually equivalent to Point 9.



Exercise

1. Construct the Lissajous figures for thefollowingcomponent oscillations. If you are using graphical method, you may have to take more than 9 points to get the complete graph in somecases.

   (a) x=2sint,y=cos2t
   (b) x=sint,y=cos(t+π/4)
   (c) x=sinπt,y=2sin(πt+π/2)

2. Any periodic motion, regardless of its complexity, can be reduced to the

sum of a number of simple harmonic motions by the application of the superposition principle. Is the above statement true orfalse?

3. A body is executing simple harmonic motion, and its displacement at time t is given by

$$x = 5 \sin 3\pi t$$

Plot the displacement, velocity, and acceleration for two complete periods.

4. A particle is simultaneously subjected to twosimpleharmonic motions in the same direction in accordance with the following equations:

$$y_1 = 8\sin 2\pi t \text{ and } \quad y_2 = 4\sin 6\pi t$$

Show graphically the resultant path of the particle.

5. A particle is subjected to three simple harmonic oscillations, one along the x-axis, second along the y-axis and the third along the z-axis. The three motions are given by$x = Asin\omega t$, $y = Bsin\omega t$, $z = C\ sin\omega t$ .Calculate the amplitude of the resultant motion.

6. Choose the correct one.

A particle moves on the x-axis according to the equation $x = A + Bsin\omega t$
The motion is a SHM with amplitude

(a) A (b) B (c) A+B (d) $\sqrt{A^2 + B^2}$

7. Choose the correct option.

More than one choice can be correct.
Which of the following will change the time period as they are taken to Mars?
(a) A simple pendulum (b) A compound pendulum (c)LC circuit
(d) A torsional pendulum (e) A spring-mass system

8. A particle is under the influence of two simultaneous SHMs in mutually perpendicular directions given by $x = cos\pi t,\ \ y = cos\pi t/2$ determine the trajectory of the resulting motion of the particle.