LAPLACE'S EQUATION IN SPHERICAL COORDINATES

We begin with Laplace's equation:

$$\nabla^2 V = 0 \qquad (1)$$

We can write the Laplacian in spherical coordinates as:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 V}{\partial \phi^2}\right) \qquad (2)$$

where θ is the polar angle measured down from the positive Z axis, and φ is the azimuthal angle.

Let's assume azimuthal symmetry; that means that our parameter *V* does not vary in the φ direction. In other words, $\partial V / \partial \phi = 0$,
so we can write the Laplacian in (2) a bit more simply. Assuming azimuthal symmetry, eq. (2) becomes:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) \qquad (3)$$

This is the form of Laplace's equation we have to solve if we want to find the electric potential in spherical coordinates. First, let's apply the method of separable variables to this equation to obtain a general solution of Laplace's equation, and then we will use our general solution to solve a few different problems.

To solve Laplace's equation in spherical coordinates, we write:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) = 0 \qquad (4)$$

The Trial Solution

The first step in solving partial differential equations using separable variables is to assume a solution of the form:

$$V = R(r)\Theta(\theta) \qquad (5)$$

where *R(r)* is a function only of *r*, and $\Theta(\theta)$ is a function only of θ. This means that we can set:

$$\frac{\partial V}{\partial r} = R'(r)\Theta(\theta); \quad \frac{\partial V}{\partial\theta} = R(r)\Theta'(\theta) \qquad (6)$$

Substituting the relationships in (6) into (4) produces:

$$\nabla^2 V = \frac{\Theta(\theta)}{r^2}\frac{\partial}{\partial r}(r^2 R'(r)) + \frac{R(r)}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\Theta'(\theta)) = 0 \qquad (7)$$

This further reduces to:

$$\nabla^2 V = \frac{1}{R(r)}\frac{d}{dr}(r^2 R'(r)) + \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}(\sin\theta\,\Theta'(\theta)) = 0 \qquad (8)$$

Separation of Variables

Equation (8) allows us to separate Laplace's equation into two separate ordinary differential equations; one being a function of *r* and the other a function of θ. Since it is true for every pair of *r* and θ, they are separately constant. This means we can separate (8) into:

$$\frac{1}{R(r)}\frac{d}{dr}(r^2 R'(r)) = l(l+1) \quad \textit{and} \quad \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}(\sin\theta\,\Theta'(\theta)) = -l(l+1) \qquad (9)$$

The radial equation

Let's start by solving the radial equation of eq. (9).

We multiply through by *R(r)* and expand the derivate to find:

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - l(l+1)R = 0 \qquad (10)$$

The above eqation can be reduced to a second order differential equation with constant coefficient by means of Euler's trick and the final solution we get is of the form:

$$R(r) = Ar^l + Br^{-(l+1)} \qquad (11)$$

where *A* and *B* are constants which will be determined once we apply specific boundary equations.

The angular equation

We solve the angular portion of equation (9) by multiplying through by Θ(θ).
and expanding the derivative to obtain:

$$\frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + l(l+1)\,\Theta = 0$$

This is actually Legendre's differential equation.
We know that the solutions to the Legendre differential equation (these are not the general solution) which we are looking for the problems of Electrostatics are the Legendre polynomials, P_l (cos θ).

Constructing the complete solution:

So the solution for each integer *l*:

$$V(r,\theta) = (Ar^{l} + Br^{-(l+1)})P_{l}(\cos\theta) \quad (13)$$

Thus taking sum over *l* the general solution to Laplace's equation in spherical coordinates is:

$$V(r,\theta) = \sum^{\infty} (A_{l} r^{l} + B_{l} r^{-(l+1)})P_{l}(\cos\theta) \quad (14)$$